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AN EXAMPLE OF A NON-BOREL LOCALLY-CONNECTED FINITE-DIMENSIONAL TOPOLOGICAL GROUP

According to a classical theorem of Gleason and Montgomery, every finite-dimensional locally path-connected topological group is a Lie group. In the paper for every natural number \(n\) we construct a locally connected subgroup \(G \subset \mathbb{R}^{n+1}\) of dimension \(n\), which is not locally compact. This answers a question posed by S. Maillot on MathOverflow and shows that the local path-connectedness in the result of Gleason and Montgomery can not be weakened to the local connectedness.

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By a classical result of A. Gleason [3] and D. Montgomery [6], every locally path-connected finite-dimensional topological group \(G\) is locally compact and hence is a Lie group. Generalizing this result of A. Gleason and D. Montgomery, T. Banakh and L. Zdomskyy [1] proved that a topological group \(G\) is a Lie group if \(G\) is compactly finite-dimensional and locally continuum-connected. In [5] Sylvain Maillot asked if the locally path-connectedness in the result of A. Gleason and D. Montgomery can be replaced by the local connectedness. In this paper we construct a counterexample to this question of S. Maillot.

We recall that a subset \(B\) of a Polish space \(X\) is called a Bernstein set in \(X\) if both \(B\) and \(X \setminus B\) meet every uncountable closed subset \(F\) of \(X\). Bernstein sets in Polish space can be easily constructed by transfinite induction, see [4, 8.24].

**Theorem 1.** For every \(n \geq 2\) the Euclidean space \(\mathbb{R}^n\) contains a dense additive subgroup \(G \subset \mathbb{R}^n\) such that

1) \(G\) is a Bernstein set in \(\mathbb{R}^n\);
2) \(G\) is locally connected;
3) \(G\) has dimension \(\dim(G) = n - 1\);
4) \(G\) is not Borel and hence not locally compact.

**Proof.** Let \((F_\alpha)_{\alpha < \mathfrak{c}}\) be an enumeration of all uncountable closed subsets of \(\mathbb{R}^n\) by ordinal \(< \mathfrak{c}\). Fix any point \(p \in \mathbb{R}^n \setminus \{0\}\). By transfinite induction, for every ordinal \(\alpha < \mathfrak{c}\) we shall choose a point \(z_\alpha \in F_\alpha\) such that the subgroup \(G_\alpha \subset \mathbb{R}^n\) generated by the set \(\{z_\beta\}_{\beta < \alpha}\) does not contain...
the point $p$. Assume that for some ordinal $\alpha < \kappa$ we have chosen points $z_\beta, \beta < \alpha$, so that the subgroup $G_{<\alpha}$ generated by the set $\{z_\beta\}_{\beta < \alpha}$ does not contain $p$. Consider the set

$$Z = \left\{ \frac{1}{n}(p - g) : n \in \mathbb{Z} \setminus \{0\}, g \in G_{<\alpha} \right\}$$

and observe that it has cardinality

$$|Z| \leq \omega \cdot |G_{<\alpha}| \leq \omega + |\alpha| < \kappa.$$ 

Since the uncountable closed subset $F_\alpha$ of $\mathbb{R}^n$ has cardinality $|F_\alpha| = \kappa$ (see [4, 6.5]), there is a point $z_\alpha \in F_\alpha \setminus Z$. For this point we get $p \neq nz_\alpha + g$ for any $n \in \mathbb{Z} \setminus \{0\}$, and $g \in G_{<\alpha}$. Consequently, the subgroup

$$G_\alpha = \{nz_\alpha + g : n \in \mathbb{Z}, g \in G_{<\alpha} \}$$

generated by the set $\{z_\beta\}_{\beta \leq \alpha}$ does not contain the point $p$. This completes the inductive step.

After completing the inductive construction, consider the subgroup $G$ generated by the set $\{a_\alpha\}_{\alpha < \kappa}$ and observe that $p \not\in G$ and $G$ meets every uncountable closed subset $F$ of $\mathbb{R}^n$. Moreover, since $G$ meets the closed uncountable set $F - p$, the coset $p + G \subset \mathbb{R}^n \setminus G$ meets $F$. So, both the subgroup $G$ and its complement $\mathbb{R}^n \setminus G$ meet each uncountable closed subset of $\mathbb{R}^n$, which means that $G$ is a Bernstein set in $\mathbb{R}^n$. The following proposition implies that the group $G$ has properties (2)–(4).

**Proposition 1.** Let $n \geq 2$. Every Bernstein subset $B$ of $\mathbb{R}^n$ has the following properties:

1) $B$ is not Borel;
2) $B$ is connected and locally connected;
3) $B$ has dimension $\dim(B) = n - 1$.

**Proof.** 1. By [4, 8.24], the Bernstein set $B$ is not Borel (more precisely, $B$ does not have the Baire property in $\mathbb{R}^n$).

2. To prove that $B$ is connected and locally connected, it suffices to prove that for every open subset $U \subset \mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$ the intersection $U \cap G$ is connected. Assuming the opposite, we could find two non-empty open disjoint sets $U_1, U_2 \subset U$ such that $U \cap B = (U_1 \cap B) \cup (U_2 \cap B)$. Consider the complement $F = U \setminus (U_1 \cup U_2) \subset U \setminus B$ and observe that $F$ is closed in $U$ and hence of type $F_\sigma$ in $\mathbb{R}^n$. If $F$ is uncountable, then $F$ contains an uncountable closed subset of $\mathbb{R}^n$ and hence meets the set $B$, which is not the case. So, the closed subset $F$ of $U$ is at most countable and separates the space $U \cong \mathbb{R}^n$, which contradicts Theorem 1.8.14 of [2].

3. Since the subset $B$ has empty interior in $\mathbb{R}^n$, we can apply Theorem 1.8.11 of [2] and conclude that $\dim(B) < n$. On the other hand, Lemma 1.8.16 [2] guarantees that $B$ has dimension $\dim(B) \geq n - 1$ (since $B$ meets every non-trivial compact connected subset of $\mathbb{R}^n$). So, $\dim(B) = n - 1$. 

\[\square\]


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