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SUPEREXTENSIONS OF CYCLIC SEMIGROUPS

Given a cyclic semigroup $S$ we study right and left zeros, singleton left ideals, the minimal ideal, left cancelable and right cancelable elements of superextensions $\lambda(S)$ and characterize cyclic semigroups whose superextensions are commutative.

Key words and phrases: cyclic semigroup, maximal linked system, superextensions.

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INTRODUCTION

This paper is devoted to describing the structure of superextensions of cyclic semigroups. The thorough study of algebraic properties of superextensions of semigroups was started in [1, 2, 3, 4, 10], where we focused at describing of superextensions of groups, and continued in [5, 6], where we studied the structure of superextensions of semilattices and inverse semigroups.

A family $F$ of nonempty subsets of a set $X$ that is closed under taking supersets and finite intersections is called a filter. A filter $U$ is called an ultrafilter if $U = F$ for any filter $F$ containing $U$. A family of subsets of a set $X$ is called a linked system if intersection of any two elements is nonempty. A linked system $M$ is said to be a maximal linked system if $M = \mathcal{L}$ for any linked system $\mathcal{L}$ containing $M$. The family $\beta(X)$ of all ultrafilters on a set $X$ is called the Stone-Čech compactification, and the family $\lambda(X)$ of all maximal linked systems is well-known [11, 12] as the superextension of a set $X$.

Each map $f : X \to Y$ induces a map (see [8])

$$
\lambda f : \lambda(X) \to \lambda(Y), \quad \lambda f : \mathcal{M} \mapsto \langle f(M) \subset Y : M \in \mathcal{M} \rangle.
$$

Here for a family $B$ of nonempty subsets of a set $Y$ by $\langle B \subset Y : B \in B \rangle = \{ A \subset Y : \exists B \in B \ (B \subset A) \}$, we denote the family $\langle B \subset Y : B \in B \rangle$. An ultrafilter $\langle \{ x \} \rangle$, generated by a singleton $\{ x \}$, $x \in X$, is called principal. We consider $X \subset \beta(X) \subset \lambda(X)$ if each point $x \in X$ is identified with the principal ultrafilter $\langle \{ x \} \rangle$ generated by the singleton $\{ x \}$.

It was shown in [9] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $\circ : \lambda(S) \times \lambda(S) \to \lambda(S)$ by the formula

$$
\mathcal{L} \circ \mathcal{M} = \langle \bigcup_{a \in L} a \ast M_a : L \in \mathcal{L}, \quad \{ M_a \}_{a \in L} \subset \mathcal{M} \rangle
$$


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for maximal linked systems \( \mathcal{L}, \mathcal{M} \in \lambda(S) \). In this case the Stone-Čech compactification \( \beta(S) \) is a subsemigroup of the superextension \( \lambda(S) \).

A nonempty subset \( I \) of a semigroup \( (S, \ast) \) is called an ideal (resp. a right ideal, a left ideal) if \( I \ast S \cup S \ast I \subset I \) (resp. \( I \ast S \subset I, S \ast I \subset I \)). An element \( z \) of a semigroup \( (S, \ast) \) is called a zero (resp. a left zero, a right zero) in \( S \) if \( a \ast z = z \ast a = z \) (resp. \( z \ast a = z, a \ast z = z \)) for any \( a \in S \). It is clear that \( z \in S \) is a zero (resp. a left zero, a right zero) in \( S \) if and only if the singleton \( \{z\} \) is an ideal (resp. a right ideal, a left ideal) in \( S \). An ideal \( I \subset S \) is called minimal if any ideal of \( S \) that lies in \( I \) coincides with \( I \). By analogy we define minimal left and minimal right ideals of \( S \). The union \( K(S) \) of all minimal left (right) ideals of \( S \) coincides with the minimal ideal of \( S \), see [11, theorem 2.8]. A semigroup \( (S, \ast) \) is said to be a right zeros semigroup if \( a \ast b = b \) for any \( a, b \in S \). A map \( \phi : S \to T \) between semigroups \( (S, \ast) \) and \( (T, \circ) \) is called a homomorphism if \( \phi(a \ast b) = \phi(a) \circ \phi(b) \) for any \( a, b \in S \). A homomorphism \( \phi : S \to I \) from a semigroup \( S \) into an ideal \( I \subset S \) is called a retraction if \( \phi(a) = a \) for any element \( a \in I \). An element \( a \) of a semigroup \( S \) is called left cancelable (resp. right cancelable) if for any points \( x, y \in S \) the equation \( ax = ay \) (resp. \( xa = ya \)) implies \( x = y \). This is equivalent to saying that the left (resp. right) shift \( l_a : S \to S, l_a : x \mapsto a \ast x \) (resp. \( r_a : S \to S, r_a : x \mapsto x \ast a \)) is injective. A semigroup \( S \) is called left (right) cancellative if all elements of \( S \) are left (right) cancelable. A semigroup that is both left and right cancellative is said to be cancellative.

A semigroup \( \langle a \rangle = \{a^n\}_{n \in \mathbb{N}} \) generated by a single element \( a \) is called cyclic. If a cyclic semigroup is infinite, then it is isomorphic to the additive semigroup \( \mathbb{N} \). A finite cyclic semigroup \( S = \langle a \rangle \) also has very simple structure (see [7]). There are positive integer numbers \( r \) and \( m \) called the index and the period of \( S \) such that: (i) \( S = \langle a, a^2, \ldots, a^{m+r-1} \rangle \) and \( m + r - 1 = |S| \); (ii) for any \( i, j \in \omega \) the equality \( a^{r+i} = a^{r+j} \) holds if and only if \( i \equiv j \mod m \); (iii) \( C_m = \{a^r, a^{r+1}, \ldots, a^{m+r-1}\} \) is the minimal ideal, a cyclic and maximal subgroup of \( S \) with the neutral element \( e = a^m \in C_m \), where \( m \) divides \( n \).

From now on we denote by \( C_{r,m} \) a finite cyclic semigroup of index \( r \) and period \( m \), and maximal subgroup of \( C_{r,m} \) is denoted by \( C_m \).

1 Homomorphisms, right, left zeros and minimal (left) ideals

**Proposition 1.1.** For any homomorphism \( \phi : S \to T \) between semigroups \( (S, \ast_1) \) and \( (T, \ast_2) \) the induced map \( \lambda \phi : \lambda(S) \to \lambda(T) \) is a homomorphism of the semigroups \( (\lambda(S), \circ_1) \) and \( (\lambda(T), \circ_2) \).

**Proof.** Given two maximal linked systems \( \mathcal{L}, \mathcal{M} \in \lambda(S) \) observe that

\[
\lambda \phi(\mathcal{L} \circ_1 \mathcal{M}) = \lambda \phi\left( \bigcup_{x \in L} x \ast_1 M_x : L \in \mathcal{L}, \{M_x\}_{x \in L} \subset \mathcal{M} \right)
\]

\[
= \langle \phi\left( \bigcup_{x \in L} x \ast_1 M_x \right) : L \in \mathcal{L}, \{M_x\}_{x \in L} \subset \mathcal{M} \rangle
\]

\[
= \langle \bigcup_{x \in L} \phi(x) \ast_2 \phi(M_x) : L \in \mathcal{L}, \{M_x\}_{x \in L} \subset \mathcal{M} \rangle
\]

\[
= \langle \bigcup_{x \in \phi(L)} x \ast_2 \phi(M_x) : L \in \mathcal{L}, \{\phi(M_x)\}_{x \in \phi(L)} \subset \lambda \phi(\mathcal{M}) \rangle
\]

\[
= \langle \phi(L) : L \in \mathcal{L} \rangle \circ_2 \langle \phi(M) : M \in \mathcal{M} \rangle = \lambda \phi(\mathcal{L}) \circ_2 \lambda \phi(\mathcal{M}).
\]

\(\square\)
Let us note that for a subsemigroup $T$ of a semigroup $S$ the homomorphism $i : \lambda(T) \to \lambda(S)$, $i : A \to \langle A \rangle_S$ is injective, and thus we can identify the semigroup $\lambda(T)$ with the subsemigroup $i(\lambda(T)) \subset \lambda(S)$.

**Lemma 1.1.** Let $I$ be an ideal of a semigroup $S$. If a map $\varphi : S \to I$ is a retraction, then the map $\lambda \varphi : \lambda(S) \to \lambda(I)$ is a retraction too.

**Proof.** Indeed, let $A \in \lambda(I)$, $M \in \lambda(S)$, then $A \circ M = \left\langle \bigcup_{a \in A} a * M_a : A \in A, A \subset I, \{M_a\}_{a \in A} \subset M \right\rangle \in \lambda(I)$. By analogy $M \circ A \in \lambda(I)$, and therefore $\lambda(I)$ is an ideal of the semigroup $\lambda(S)$. If $A \in \lambda(I)$, then $\lambda \varphi(A) = \langle \varphi(A) : A \subset I, A \in A \rangle = \langle A \subset I : A \in A \rangle = A$ and hence $\lambda \varphi$ is a retraction. \[\square\]

**Lemma 1.2.** Let $I$ be an ideal of a semigroup $S$ and a map $\varphi : S \to I$ is a retraction. The semigroup $S$ has a right (left) zero if and only if the semigroup $I$ has a right (left) zero, and all right and left zeros of the semigroup $S$ are contained in $I$.

**Proof.** Let $z$ be a right (left) zero of the semigroup $S$, that is $sz = z$ ($zs = z$) for any $s \in S$. Since $\varphi$ is a homomorphism, $\varphi(s) \varphi(z) = \varphi(z)$ ($\varphi(z) \varphi(s) = \varphi(z)$). Specifically for any $s \in I$ the equality $\varphi(s) = s$ holds, and then $s \varphi(z) = \varphi(s) \varphi(z) = \varphi(z)$ ($\varphi(z) s = \varphi(z) \varphi(s) = \varphi(z)$). Consequently, $\varphi(z)$ is a right (left) zero of the semigroup $I$.

Let $z \in I$ be a right (left) zero of the semigroup $I$. Since $I$ is an ideal, then for any $s \in S$ we have that $sz, zs \in I$, and hence $sz = \varphi(sz) = \varphi(s) \varphi(z) = \varphi(s) z = z (zs = \varphi(zs) = \varphi(z) \varphi(s) = z \varphi(s) = z)$. Consequently, $z$ is a right (left) zero of the semigroup $S$.

If $z$ is a right (left) zero of the semigroup $S$, then $z = sz \in I$ ($z = zs \in I$), where $s \in I$. Therefore, all right (left) zeros of the semigroup $S$ are contained in $I$. \[\square\]

Let $e$ be the neutral element of the maximal subgroup $C_m$ of a cyclic semigroup $C_{r,m}$.

**Lemma 1.3.** The map $\varphi : C_{r,m} \to C_m, \varphi(x) = ex$ is a retraction and $\varphi(x)y = xy$ for any $x \in C_{r,m}$ and $y \in C_m$.

**Proof.** Since the semigroup $C_m$ is an ideal of the semigroup $C_{r,m}$, $\varphi(x) = ex \in C_m$. Consequently, $\varphi(xy) = exy = exy = exy = \varphi(x) \varphi(y)$ for any $x, y \in C_{r,m}$ and $\varphi(x) = ex = x$ for $x \in C_m$. Hence the map $\varphi : C_{r,m} \to C_m$ is a retraction. Further for any $x \in C_{r,m}$ and $y \in C_m$ we have that $xy \in C_m$, and therefore $\varphi(xy) = xy$. On the other hand, $\varphi(xy) = \varphi(x) \varphi(y) = \varphi(x)y$, since $y \in C_m$. \[\square\]

Combining Lemmas 1.1–1.3 we get

**Proposition 1.2.** The semigroup $\lambda(C_{r,m})$ contains a right (left) zero if and only if its subgroup $\lambda(C_m)$ contains a right (left) zero. Each right (left) zero of $\lambda(C_{r,m})$ belongs to $\lambda(C_m)$.

It was proved in [1] that the semigroup $\lambda(G)$ possesses a right zero if and only if the group $G$ is periodic and each element of $G$ has odd order. Since each element of a finite group $G$ has odd order if and only if the group $G$ has odd order, Proposition 1.2 implies the following characterization of superextensions of finite cyclic semigroups that have right zeros.

**Theorem 1.** The superextension $\lambda(C_{r,m})$ of a finite cyclic semigroup $C_{r,m}$ has a right zero if and only if the period $m$ of the cyclic semigroup $C_{r,m}$ is an odd number.
Proposition 1.3. The superextension of the infinite cyclic semigroup has neither right nor left zeros.

Proof. Let \( \langle a \rangle = \{a, a^2, \ldots, a^n \ldots \} \) be the infinite cyclic semigroup and \( M \in \lambda(\langle a \rangle) \). First observe that if \( \langle a \rangle = A \cup B \) is any partition of the set \( \langle a \rangle \), then either \( A \in M \) or \( B \in M \). Indeed, if \( A \notin M \), then \( M \cap B \neq \emptyset \) for any \( M \in M \), and thus the maximality of \( M \) implies that \( B \in M \). Consider the partition \( \langle a \rangle = A \cup B \), where \( A = \{a, a^3, \ldots, a^{2k-1}, \ldots \} \), \( B = \{a^2, a^4, \ldots, a^{2k}, \ldots \} \). Assume that a maximal linked system \( M \) is a right (left) zero of the semigroup \( \langle a \rangle \). Then for any \( x \in \langle a \rangle \) we have \( \langle \{x\} \rangle \circ M = M (M \circ \langle \{x\} \rangle = M) \), and therefore \( xM \in M (Mx \in M) \) for any \( M \in M \). If \( A \in M \), then \( B = aA = Aa \in M \), that is impossible, since \( A \cap B = \emptyset \). By analogy, if \( B \in M \), then \( A \cap aB = Ba \in M \). This contradiction implies that the superextension of the infinite cyclic semigroup contains neither right nor left zeros. \( \square \)

It was proved in [1] that for the semigroup \( \lambda(G) \) has a (left) zero if and only if a group \( G \) is of order \( |G| \in \{1,3,5\} \).

Consequently, Proposition 1.2 implies the following characterization of superextensions of finite cyclic semigroups that have (left) zeros.

Theorem 2. The superextension \( \lambda(C_{r,m}) \) of a cyclic semigroup \( C_{r,m} \) has a (left) zero if and only if \( m \in \{1,3,5\} \).

Now we shall characterize cyclic semigroups whose superextensions have one-point minimal left ideals.

If \( C_{r,m} \) is a finite cyclic semigroup of odd period \( m \) and \( C_n \) is the maximal subgroup of \( C_{r,m} \), then the superextension \( \lambda(C_{r,m}) \) contains a right zero, in particular the maximal linked system

\[
\mathcal{L} = \{A \subset C_m : |A| > m/2\}
\]

is a right zero of the semigroup \( \lambda(C_{r,m}) \). A maximal linked system \( Z \in \lambda(C_{r,m}) \) is a right zero of the semigroup \( \lambda(C_{r,m}) \) if and only if the one-point set \( \{Z\} \) is a minimal left ideal of \( \lambda(C_{r,m}) \). Taking into account that all minimal left ideals are isomorphic and the union \( K(\lambda(C_{r,m})) \) of all minimal left ideals in \( \lambda(C_{r,m}) \) coincides with the minimal ideal of \( \lambda(C_{r,m}) \) (see [11, Theorem 2.8]), Theorem 1 and Proposition 1.3 imply the following theorem.

Theorem 3. A finite cyclic semigroup \( C_{r,m} \) has odd period \( m \) if and only if all minimal left ideals of the semigroup \( \lambda(C_{r,m}) \) are singletons. In this case the minimal ideal \( K(\lambda(C_{r,m})) \) of the semigroup \( \lambda(C_{r,m}) \) is the subsemigroup of right zeros of \( \lambda(C_{r,m}) \). The infinite cyclic semigroup has no one-point minimal left (right) ideals.

2 Commutativity of superextensions of cyclic semigroups

Theorem 4. A finite cyclic semigroup \( C_{r,m} = \{a, a^2, \ldots, a^r, \ldots, a^{m+r-1} | a^{r+m} = a^r \} \) of order \( m + r - 1 \) has commutative superextension if and only if

\[
(r, m) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}.
\]

The superextension of the infinite cyclic semigroup is not commutative.
Proof. It was proved in the paper [1] that the superextension of a group $G$ is commutative if and only if $|G| \leq 4$. Since for $m > 4$ the superextension $\lambda(C_{r,m})$ contains a noncommutative subsemigroup $\lambda(C_m)$, $\lambda(C_{r,m})$ is not commutative. So it is sufficient to consider only cyclic semigroups of period $m \leq 4$.

If index $r = 1$, then $C_{r,m}$ is a cyclic group of order $m$, and thus for $r = 1$ the semigroup $\lambda(C_{r,m})$ is commutative if and only if $m \leq 4$.

If $|C_{r,m}| \in \{1, 2\}$, then the superextension $\lambda(C_{r,m})$ is isomorphic to the semigroup $C_{r,m}$, and $\lambda(C_{r,m})$ is commutative. In the case $|C_{r,m}| = 3$ the superextension $\lambda(C_{r,m})$ contains only one maximal linked system, which is not a principal ultrafilter. Since all principal ultrafilters commute with maximal linked systems, the superextension $\lambda(C_{r,m})$ is commutative.

It follows that for

$$(r, m) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (3, 1)\}$$

the superextension $\lambda(C_{r,m})$ is commutative.

If $r = 2$, $m \in \{3, 4\}$, then the product $xy$ of any two elements $x, y \in C_{r,m}$ is contained in the maximal subgroup $C_m$, and thus $xy = \varphi(xy) = \varphi(x)\varphi(y)$, where $\varphi : C_{r,m} \to C_m$ is the retraction $\varphi : s \to es$. Since superextensions of groups of order 3 and 4 are commutative,

$$(A \circ B = \lambda\varphi(A) \circ \lambda\varphi(B) = \lambda\varphi(B) \circ \lambda\varphi(A) = B \circ A)$$

for any $A, B \in \lambda(C_{r,m})$. Consequently, the semigroups $\lambda(C_{2,3})$ and $\lambda(C_{2,4})$ are commutative.

Let $r = 3$. The case $m = 1$ was considered before.

For the semigroup $C_{3,2} = \{a, a^2, a^3, a^4 | a^5 = a^3\}$ the semigroup $\lambda(C_{3,2})$ contains 12 elements:

$${\mathcal{U}}_k = \langle \{a^k\} \rangle, \quad \Delta_k = \langle A \subset C_{3,2} : |A| = 2, \ a^k \notin A \rangle$$

and

$$\square_k = \langle C_{3,2} \setminus \{a^k\}, A : A \subset C_{3,2}, |A| = 2, \ a^k \in A \rangle, \text{ where } k \in \{1, 2, 3, 4\}.$$

The following table implies the commutativity of $\lambda(C_{3,2})$:

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If $m \in \{3, 4\}$, then $C_{3,m} = \{a, a^2, \ldots, a^{m+2} | a^{m+3} = a^3\}$. Consider maximal linked systems $A = \langle \{a, a^2\}, \{a, a^3\}, \{a^2, a^3\} \rangle$ and $B = \langle \{a, a^2\}, \{a, a^{m+1}\}, \{a^2, a^{m+1}\} \rangle$. Observe that $\{a^2, a^3\} = a\{a, a^2\} \cup a^2\{a, a^{m+1}\} \in A \circ B$, but $\{a^2, a^3\} \notin B \circ A$. Therefore, $A \circ B \neq B \circ A$ and the semigroup $C_{3,m}$ is not commutative.

Let $r \geq 4$. First consider the case of the semigroup $C_{4,1} = \{a, a^2, a^3, a^4 | a^5 = a^4\}$. Each maximal linked system different from the principal ultrafilter $\langle \{a\} \rangle$ contains the set $\{a^2, a^3, a^4\}$. 


Since \( \{a^2, a^3, a^4\} \) is an ideal in \( C \), the product of such maximal linked systems is the principal ultrafilter \( \{a^4\} \). The fact that the principal ultrafilter \( \{a\} \) commutes with all maximal linked systems implies the commutativity of the semigroup \( \lambda(C_{4,1}) \).

Put \( A = \{a, a^2\}, \{a, a^3\}, \{a^2, a^3\}, B = \{a, a^2\}, \{a, a^m\}, \{a^2, a^m\} \). We have that \( a^2, a^3 = a\{a^2, a^3\} \cup a^2\{a, a^2\} \in B \circ A \), and \( \{a^2, a^3\} \notin A \circ B \), since the equality \( a^{m+r+1} = a^4 \) holds only if \( r = 4 \) and \( m = 1 \), which we considered before. Consequently, \( A \circ B \neq B \circ A \) and a semigroup \( \lambda(C_{r,m}) \) for \( (r, m) \neq (4, 1) \) is not commutative.

Let \( \{a\} = \{a, \ldots, a^n, \ldots\} \) be the infinite cyclic semigroup. Put \( A = \{a, a^2\}, \{a, a^3\}, \{a^2, a^3\}, B = \{a, a^2\}, \{a, a^3\}, \{a^2, a^3\} \). Let us observe that \( \{a^2, a^3\} = a\{a^2, a^3\} \cup a^2\{a, a^2\} \in B \circ A \), but \( \{a^2, a^3\} \notin A \circ B \). Therefore, \( A \circ B \neq B \circ A \) and the semigroup \( \lambda(\{a\}) \) is not commutative. 

### 3 Right (left) cancelable elements

In this section we shall detect right (left) cancelable elements of superextensions of cyclic semigroups.

**Proposition 3.1.** The superextension \( \lambda(C_{r,m}) \) has (left, right) cancelable elements if and only if index \( r \) of a cyclic semigroup \( C_{r,m} \) is equal to 1.

**Proof.** Let \( r > 1 \) and \( a \) be the generator of a semigroup \( C_{r,m} \). Consider the map \( \varphi : C_{r,m} \to C_m \), \( \varphi : x \to ex \), where \( e \) is the neutral element of the cyclic group \( C_m \). According to Lemma 1.3 this map is a retraction. Since \( a^{r-1}x \in C_{r,m} = \{a^{r-1}, \ldots, a^{r+m-1}\} \) for any \( x \in C_{r,m} \), \( a^{r-1}x = \varphi(a^{r-1}x) = \varphi(a^{r-1})\varphi(x) \). On the other hand, since \( C_m \) is an ideal of \( C_{r,m} \), \( \varphi(a^{r-1})x \in C_m \) and \( \varphi(a^{r-1})x = \varphi(\varphi(a^{r-1})x) = \varphi(\varphi(a^{r-1}))\varphi(x) = \varphi(a^{r-1})\varphi(x) \). Consequently, \( \varphi(a^{r-1})x = a^{r-1}x \) for any \( x \in C_{r,m} \).

Let \( M \) be a maximal linked system on a semigroup \( C_{r,m} \). Then we obtain \( \{a^{r-1}\} \circ M = \{ \bigcup_{a \in \{a^{r-1}\}} a \circ M : \{M_a\}_{a \in L} \subset M \} = \{a^{r-1}M : M \in M \} = \{\varphi(a^{r-1})M : M \in M \} = \{\{\varphi(a^{r-1})\} \circ M \) and \( M \circ \{a^{r-1}\} = \{\bigcup_{a \in M} a \circ \{a^{r-1}\} : M \in M \} = \{M \varphi(a^{r-1}) : M \in M \} = \{M \circ \{a^{r-1}\} \} \). Since \( a^{r-1} \neq \varphi(a^{r-1}) \), the maximal linked system \( M \) is neither left nor right cancelable.

If \( r = 1 \), then a cyclic semigroup \( C_{1,m} = C_m \) is a group. Let \( e \) be the neutral element of the group \( C_m \). Then \( \{e\} \circ M = M = \{e\} \) for any \( M \in \lambda(C_m) \), and equalities \( X \circ \{e\} = M = \{e\} \circ Y \) imply that \( X = Y \). Consequently, the principal ultrafilter \( \{e\} \) is a cancelable element of the semigroup \( \lambda(C_{1,m}) \).

If \( G \) is a group, then the formula

\[
\mathcal{L} \circ M = \bigcup_{a \in L} a \circ M_a : L \in \mathcal{L}, \{M_a\}_{a \in L} \subset M
\]

implies that the product \( \mathcal{L} \circ M \) of any two maximal linked systems \( \mathcal{L} \) and \( M \) is a principal ultrafilter if and only if both \( \mathcal{L} \) and \( M \) are principal ultrafilters. Therefore, we deduce the following proposition.

**Proposition 3.2.** For a group \( G \) the set \( \lambda(G) \) \( \{ \{g\} : g \in G \} \) is an ideal in \( \lambda(G) \).

**Lemma 3.1.** A semigroup \( S \) is a left (right) cancellative semigroup if and only if all principal ultrafilters are left (right) cancelable elements in the superextension \( \lambda(S) \).
Proof. If an element \( a \in S \) is not left (right) cancelable in the semigroup \( S \), then it is clear that the principal ultrafilter generated by the element \( a \) is not cancelable in \( \lambda(S) \).

Let \( S \) be a left (right) cancellative semigroup, \( a \in S \) and \( \mathcal{X}, \mathcal{Y} \subseteq \lambda(S) \), \( \mathcal{X} \neq \mathcal{Y} \), then \( \mathcal{X} \cap \mathcal{Y} = \emptyset \) for some \( X \in \mathcal{X}, Y \in \mathcal{Y} \). Since each element of \( S \) is left (right) cancelable, then \( aX \cap aY = \emptyset \) \( (Xa \cap Ya = \emptyset) \), and thus \( \langle \{a\} \rangle \circ \mathcal{X} \neq \langle \{a\} \rangle \circ \mathcal{Y} \) \( (\mathcal{X} \circ \langle \{a\} \rangle \neq \mathcal{Y} \circ \langle \{a\} \rangle) \). Consequently, the left \( l\langle \{a\} \rangle \) (right \( r\langle \{a\} \rangle \)) shift is injective and the principal ultrafilter \( \langle \{a\} \rangle \) is left (right) cancelable.

**Proposition 3.3.** An element \( M \in \lambda(C_{1,m}) \) is left (right) cancelable if and only if \( M \) is a principal ultrafilter.

Proof. Since in any group, in particular in the cyclic group \( C_{1,m} \), all elements are cancelable, according to Lemma 3.1 all principal ultrafilters are right cancelable in the superextension \( \lambda(C_{1,m}) \).

Assume that some maximal linked system \( M \in \lambda(C_{1,m}) \setminus \{\langle g \rangle \} \) \( g \in C_{1,m} \) is left cancelable. This means that the left shift \( l_M : \lambda(C_{1,m}) \to \lambda(C_{1,m}) \), \( l_M : A \to M \ast A \), is injective. According to Theorem 3.2, the set \( \lambda(C_{1,m}) \setminus \{\langle g \rangle \} \) \( g \in C_{1,m} \) is an ideal in \( \lambda(C_{1,m}) \). Consequently, \( l_M(\lambda(C_{1,m})) = M \circ \lambda(C_{1,m}) \) \( \subseteq \lambda(C_{1,m}) \setminus \{\langle g \rangle \} \) \( g \in C_{1,m} \). Since \( \lambda(C_{1,m}) \) is finite, \( l_M \) cannot be injective.

For the right cancelable elements the proof is analogous.

Since the infinite cyclic semigroup is a cancellative semigroup, then Lemma 3.1 implies the following proposition.

**Proposition 3.4.** All principal ultrafilters are cancelable elements in the superextension of the infinite cyclic semigroup.

**Proposition 3.5.** Let \( S \) be the infinite cyclic semigroup and \( \mathcal{L} \subseteq \lambda(S) \). A maximal linked system \( \mathcal{L} \) is right cancelable in \( \lambda(S) \) provided for every \( s \in S \) there is a set \( L_s \subseteq \mathcal{L} \) such that the family \( \{s \ast L_s : s \in S\} \) is disjoint.

Proof. Assume that \( \{L_s\}_{s \in S} \subset \mathcal{L} \) is a family such that \( \{s \ast L_s : s \in S\} \) is disjoint. To prove that \( \mathcal{L} \) is right cancelable, take two maximal linked systems \( \mathcal{A}, \mathcal{B} \subseteq \lambda(S) \) with \( \mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L} \). It is sufficient to show that \( \mathcal{A} \subseteq \mathcal{B} \). Take any set \( A \in \mathcal{A} \) and observe that the set \( \bigcup_{a \in A} a \ast L_a \) belongs to \( \mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L} \). Consequently, there is a set \( B \subseteq \mathcal{B} \) and a family of sets \( \{M_b\}_{b \in B} \subseteq \mathcal{L} \) such that

\[
\bigcup_{b \in B} b \ast M_b \subset \bigcup_{a \in A} a \ast L_a.
\]

It follows from \( L_b \in \mathcal{L} \) that \( M_b \cap L_b \) is not empty for every \( b \in B \).

Since the sets \( a \ast L_a \) \( b \ast L_b \) are disjoint for different \( a, b \in S \), the inclusion

\[
\bigcup_{b \in B} b \ast (M_b \cap L_b) \subset \bigcup_{b \in B} b \ast M_b \subset \bigcup_{a \in A} a \ast L_a
\]

implies \( B \subseteq \mathcal{A} \) and hence \( \mathcal{A} \subseteq \mathcal{B} \).
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Ключові слова і фрази: циклічна напівгрупа, максимальна зцеплена система, суперрозширення.


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