THE COMPLETENESS OF A NORMED SPACE IS EQUIVALENT TO THE
HOMOGENEITY OF ITS SPACE OF CLOSED BOUNDED CONVEX SETS

We prove that an infinite-dimensional normed space $X$ is complete if and only if the space $B_{\text{Conv}}^H(X)$ of all non-empty bounded closed convex subsets of $X$ is topologically homogeneous.

Key words and phrases: completeness, normed spaces, topological homogeneity, closed convex sets.

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INTRODUCTION

In this paper we shall prove that the completeness of an infinite-dimensional normed space $X$ is equivalent to the topological homogeneity of its hyperspace $B_{\text{Conv}}^H(X)$ of all non-empty bounded closed convex sets. The space $B_{\text{Conv}}^H(X)$ is endowed with the Hausdorff metric

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}, \quad A, B \in B_{\text{Conv}}^H(X).$$

Due to results of [5], [6], [2], the topological structure of the hyperspace $B_{\text{Conv}}^H(X)$ is well-understood for each Banach space $X$. To formulate a classification result for the hyperspace $B_{\text{Conv}}^H(X)$ we need to recall some notations.

All linear spaces considered in this paper are over the field of real numbers $\mathbb{R}$. For a linear topological space $X$ its dimension $\dim(X)$ is defined as the smallest cardinality $|B|$ of a subset $B \subset X$ having dense linear hull in $X$. For a cardinal $\kappa$ by $l_2(\kappa) = \{ x \in \mathbb{R}^\kappa : \sum_{\alpha \in \kappa} |x(\alpha)|^2 < \infty \}$ we denote the Hilbert space having an orthonormal base of cardinality $\kappa$. By $\omega$ we denote the smallest infinite cardinal. By $\mathbb{R}_+$ and $\mathbb{I}$ we denote the closed half-line $[0, \infty)$ and the closed unit interval $[0, 1]$, respectively.

The following classification theorem can be derived from [5], [6], [2].

**Theorem 1.** For each Banach space $X$ the hyperspace $B_{\text{Conv}}^H(X)$ is homeomorphic to:

1) \{0\} iff $\dim(X) = 0$;
2) $\mathbb{R}_+ \times \mathbb{R}$ iff $\dim(X) = 1$;
3) $\mathbb{I}^\omega \times \mathbb{R}_+$ iff $1 < \dim(X) < \omega$;
4) $l_2(2^{\dim(X)})$ iff $\dim(X) \geq \omega$.

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In this paper we shall study the hyperspace \( \mathrm{BConv}_H(X) \) for non-complete normed spaces \( X \). In this case we shall show that \( \mathrm{BConv}_H(X) \) has rather bad topological properties. In particular, it is neither topologically homogeneous nor even weakly homogeneous.

1 Main Result

A topological space \( X \) is defined to be

- **topologically homogeneous** if for any two points \( x, y \in X \) there is a homeomorphism \( h : X \to X \) such that \( h(x) = y \);

- **weakly homogeneous** if for each non-empty open dense subset \( U \subset X \) and each point \( x \in X \) there is a homeomorphism \( h : X \to X \) such that \( h(x) \in U \).

It is clear that each topologically homogeneous space is weakly homogeneous.

The main result of this note is the following theorem.

**Theorem 2.** For an infinite-dimensional normed space \( X \) the following conditions are equivalent:

1. \( X \) is complete;
2. \( \mathrm{BConv}_H(X) \) is topologically homogeneous;
3. \( \mathrm{BConv}_H(X) \) is weakly homogeneous;
4. \( \mathrm{BConv}_H(X) \) is homeomorphic to \( l_2(2^{\dim(X)}) \).

**Proof.** We shall prove the following implications. (1) \( \Rightarrow \) (4) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1). The implication (1) \( \Rightarrow \) (4) follows from Theorem 1 while (4) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are trivial. So, it remains to prove (3) \( \Rightarrow \) (1).

In the space \( \mathrm{BConv}_H(X) \) consider the open dense subspace

\[
\mathrm{BCb}_H(X) = \{ A \in \mathrm{BConv}_H(X) : \text{Int}(A) \neq \emptyset \}
\]

consisting of bounded convex bodies (i.e., bounded convex sets with non-empty interior). Let \( \bar{X} \) be the completion of the normed space \( X \) and \( \mathrm{BCb}_H(\bar{X}) \) be the space of bounded convex bodies in the Banach space \( \bar{X} \). Observe that the map

\[
\text{cl} : \mathrm{BCb}_H(X) \to \mathrm{BCb}_H(\bar{X}),
B \mapsto \bar{B},
\]

is an isometric bijection. The space \( \mathrm{BCb}_H(\bar{X}) \), being open in the complete metric space \( \mathrm{BConv}_H(\bar{X}) \), is \( \varepsilon \)-Cech-complete and so is its isometric copy \( \mathrm{BCb}_H(X) \). Assuming that the space \( \mathrm{BConv}_H(X) \) is weakly homogeneous, and taking into account that \( \mathrm{BCb}_H(X) \) is an open dense \( \varepsilon \)-Cech-complete subspace of \( \mathrm{BConv}_H(X) \), we conclude that each point of the space \( \mathrm{BConv}_H(X) \) has an open \( \varepsilon \)-Cech-complete neighborhood. By a result of Arhangel'ski [1] and Frolik [4] (see also [3, 5.5.8(c)]), the space \( \mathrm{BConv}_H(X) \), being locally \( \varepsilon \)-Cech-complete and paracompact, is \( \varepsilon \)-Cech-complete, and so is its closed subspace \( X \). Being \( \varepsilon \)-Cech-complete, the space \( X \) is a \( G_{32} \)-set in its completion \( \bar{X} \). Assuming that \( X \neq \bar{X} \), we can find a point \( x \in \bar{X} \setminus X \) and conclude that \( X \) and \( X + x \) are two disjoint dense \( G_{32} \)-subsets of Banach space \( \bar{X} \), which is impossible according to the Baire Theorem. Consequently, \( X = \bar{X} \) is a Banach space. \( \square \)
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REFERENCES


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