METRIC ON THE SPECTRUM OF THE ALGEBRA OF ENTIRE SYMMETRIC FUNCTIONS OF BOUNDED TYPE ON THE COMPLEX $L_\infty$

It is known that every complex-valued homomorphism of the Fréchet algebra $H_{bs}(L_\infty)$ of all entire symmetric functions of bounded type on the complex Banach space $L_\infty$ is a point-evaluation functional $\delta_x$ (defined by $\delta_x(f) = f(x)$ for $f \in H_{bs}(L_\infty)$) at some point $x \in L_\infty$. Therefore, the spectrum (the set of all continuous complex-valued homomorphisms) $M_{bs}$ of the algebra $H_{bs}(L_\infty)$ is one-to-one with the quotient set $L_\infty/\sim$, where an equivalence relation "$\sim$" on $L_\infty$ is defined by $x \sim y \Leftrightarrow \delta_x = \delta_y$. Consequently, $M_{bs}$ can be endowed with the quotient topology. On the other hand, $M_{bs}$ has a natural representation as a set of sequences which endowed with the coordinate-wise addition and the quotient topology forms an Abelian topological group. We show that the topology on $M_{bs}$ is metrizable and it is induced by the metric $d(\xi, \eta) = \sup_n \sqrt{\sum |\xi_n - \eta_n|^2}$, where $\xi = \{\xi_n\}_{n=1}^\infty, \eta = \{\eta_n\}_{n=1}^\infty \in M_{bs}$.

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INTRODUCTION

Symmetric functions on Banach spaces were studied by a number of authors [1, 3–8, 10, 12, 13] (see also a survey [2]). In particular, symmetric polynomials and symmetric analytic functions on $L_\infty$ (see definition below) were studied in [6, 12, 13].

Let $L_\infty$ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions $x$ on $[0, 1]$ with norm $\|x\|_\infty = \text{ess sup}_{t \in [0,1]} |x(t)|$.

Let $\Xi$ be the set of all measurable bijections of $[0, 1]$ that preserve the measure. A function $f : L_\infty \to \mathbb{C}$ is called symmetric if $f(x \circ \sigma) = f(x)$ for every $x \in L_\infty$ and for every $\sigma \in \Xi$.

Let $H_{bs}(L_\infty)$ be the Fréchet algebra of all entire symmetric functions $f : L_\infty \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. By [6, Theorem 4.3], polynomials $R_n : L_\infty \to \mathbb{C}, R_n(x) = \int_{[0,1]} (x(t))^n dt$ for $n \in \mathbb{N}$ form an algebraic basis in the algebra of all symmetric continuous polynomials on $L_\infty$. Since every $f \in H_{bs}(L_\infty)$ can be described by its Taylor series of continuous symmetric homogeneous polynomials, it follows that $f$ can be uniquely represented as

$$f(x) = f(0) + \sum_{n=1}^\infty \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x).$$
Consequently, for every non-trivial continuous homomorphism $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$, taking into account $\varphi(1) = 1$, we have

$$\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1 + 2k_2 + \ldots + nk_n = n} a_{k_1, \ldots, k_n} \varphi(R_1)^{k_1} \ldots \varphi(R_n)^{k_n}.$$ 

Therefore, $\varphi$ is completely determined by the sequence of its values on $R_n : (\varphi(R_1), \varphi(R_2), \ldots)$. By the continuity of $\varphi$, the sequence $\{\sqrt{n} | \varphi(R_n)| \}^\infty_{n=1}$ is bounded. On the other hand, we have the following

**Theorem 1** ([6, Section 3]). For every sequence $\xi = \{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that $\sup_{n \in \mathbb{N}} \sqrt{n} |\xi_n| < +\infty$, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$ and $\|x_\xi\|_\infty \leq \frac{2}{M} \sup_{n \in \mathbb{N}} \sqrt{n} |\xi_n|$, where

$$M = \prod_{n=1}^{\infty} \cos \left( \frac{\pi}{2n + 1} \right).$$

(1)

Hence, for every sequence $\xi = \{\xi_n\}_{n=1}^\infty$ such that $\sup_{n \in \mathbb{N}} \sqrt{n} |\xi_n| < +\infty$, there exists the point-evaluation functional $\varphi = \delta_{x_\xi}$ such that $\varphi(R_n) = \xi_n$ for every $n \in \mathbb{N}$. Since every such a functional is a continuous homomorphism, it follows that the spectrum (the set of all continuous complex-valued homomorphisms) of the algebra $H_{bs}(L_\infty)$, which we denote by $M_{bs}$, can be identified with the set of all sequences $\xi = \{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that $\{\sqrt{n} |\xi_n|\}_{n=1}^\infty$ is bounded.

Let $\nu : L_\infty \to M_{bs}$ be defined by

$$\nu(x) = (R_1(x), R_2(x), \ldots).$$

Let $\tau_\infty$ be the topology on $L_\infty$, generated by $\| \cdot \|_\infty$. Let us define an equivalence relation on $L_\infty$ by $x \sim y \iff \nu(x) = \nu(y)$. Let $\tau$ be the quotient topology on $M_{bs}$:

$$\tau = \{\nu(V) : V \in \tau_\infty\}.$$

Note that $\nu$ is a continuous open mapping.

The operation of coordinate-wise addition $+: M_{bs}^2 \to M_{bs}$ is defined by

$$a + b = (a_1 + b_1, a_2 + b_2, \ldots)$$

for $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in M_{bs}$. In [13] it is shown that $(M_{bs}, +, \tau)$ is an Abelian topological group. In this work we show that $(M_{bs}, \tau)$ is a metrizable topological space. Also we explicitly construct the metric which induces $\tau$.

1 The Main Result

Let us denote $B(x, r)$ the open ball of radius $r$ and center $x$ in $L_\infty$.

**Proposition 1.** The identity element $0 = (0, 0, \ldots)$ of the topological group $(M_{bs}, +, \tau)$ has a countable local basis of neighborhoods.

**Proof.** For $n \in \mathbb{N}$ let $U_n = \nu(B(0, \frac{1}{n}))$. Since $\nu$ is an open mapping, it follows that $U_n \in \tau$. Note that $0 \notin U_n$. Thus, $U_n$ is an open neighborhood of $0$ for every $n \in \mathbb{N}$. Let us show that a family $\{U_n : n \in \mathbb{N}\}$ form a local basis of neighborhoods of $0$. Let $W \subset M_{bs}$ be an arbitrary open neighborhood of $0$. Then $\nu^{-1}(W)$ is open in $L_\infty$ and $\nu^{-1}(W)$ contains $0$. Therefore, there exists $r > 0$ such that $B(0, r) \subset \nu^{-1}(W)$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < r$. Then $B(0, \frac{1}{n}) \subset B(0, r) \subset \nu^{-1}(W)$. Therefore, $\nu(B(0, \frac{1}{n})) \subset W$, i.e. $U_n \subset W$.  \[ \square \]
We will use Birkhoff-Kakutani theorem.

**Theorem 2** ([9, p.34]). Let $G$ be a Hausdorff topological group whose open sets at the identity element have a countable basis. Then $G$ is metrizable and, moreover, there exists a metric which is right-invariant.

**Corollary 1.** There exists an invariant metric $d$ on $M_{bs}$ which induces topology $\tau$.

**Proof.** By [13, Corollary 1], $(M_{bs}, +, \tau)$ is an Abelian topological group. By [13, Theorem 2], $\tau$ is Hausdorff. By Proposition 1, the identity element of $M_{bs}$ has a countable local basis. Therefore by Theorem 2 there exists a right-invariant metric $d$ on $M_{bs}$ which induces topology $\tau$. Since $(M_{bs}, +, \tau)$ is Abelian, the metric $d$ is also left-invariant. \hfill \Box

For $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots) \in M_{bs}$ let

$$d_I(a, b) = \sup_{n \in \mathbb{N}} \sqrt{|a_n - b_n|}.$$  

Note that analogical metric is defined on spaces of entire functions of one complex variable (where a role of sequences $a$ and $b$ play sequences of coefficients of the Taylor series of functions) and it is called Iyer metric (see e. g. [11]). Also note that a metric space $(M_{bs}, d_I)$ is isometric to the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ of the exponential type such that $f(0) = 0$ with Iyer metric.

Let $V(a, r)$ be the open ball in $M_{bs}$ of radius $r$ and center $a \in M_{bs}$ with respect to the metric $d_I$.

**Lemma 1.** Let $r > 0$ and $0 < \rho < \frac{M}{2r}$, where $M$ is defined by (1). Then $V(0, \rho) \subset v(B(0, r))$.

**Proof.** Let $a = (a_1, a_2, \ldots) \in V(0, \rho)$. Let us show that $a \in v(B(0, r))$. By Theorem 1, there exists $x_a \in L_{\infty}$ such that $v(x_a) = a$ and $\|x_a\|_{\infty} < \frac{2}{M} \sup_{n \in \mathbb{N}} \sqrt{|a_n|}$. Since $a \in V(0, \rho)$, it follows that $d_I(0, a) < \rho$, i. e. $\sup_{n \in \mathbb{N}} \sqrt{|a_n|} < \rho$. Thus, $\|x_a\|_{\infty} < \frac{2}{M} \rho$. Since $\rho < \frac{M}{2r}$, it follows that $\|x_a\|_{\infty} < r$, i. e. $x_a \in B(0, r)$. Therefore $v(x_a) \in v(B(0, r))$, i. e. $a \in v(B(0, r))$. \hfill \Box

**Theorem 3.** The metric $d_I$ induces the topology $\tau$.

**Proof.** Since both metrics $d_I$ and $d$ (given by Corollary 1) are invariant with respect to translations (in the sense that $d(a + c, b + c) = d(a, b)$ for every $a, b, c \in M_{bs}$), it suffices to prove that every open neighborhood of 0 with respect to $\tau$ contains some open ball with center 0 with respect to $d_I$ and vice versa.

Let $W \in \tau$ such that $0 \in W$. Then $v^{-1}(W)$ is the open neighborhood of 0 in $L_{\infty}$. Therefore, there exists $r > 0$ such that $B(0, r) \subset v^{-1}(W)$. By Lemma 1, for $0 < \rho < \frac{2r}{M}$ we have $V(0, \rho) \subset v(B(0, r))$. Since $v(B(0, r)) \subset W$, it follows that $V(0, \rho) \subset W$.

Let us show that for every open ball $V(0, r)$ there exists $W \in \tau$ such that $0 \in W$ and $W \subset V(0, r)$. Set $W = v(B(0, r))$. Let us show that $W \subset V(0, r)$. It suffices to prove that $v(x) \in V(0, r)$ for every $x \in B(0, r)$. For $x \in B(0, r)$ we have $\|x\|_{\infty} < r$ and, consequently,

$$|R_n(x)| \leq \|x\|_{\infty}^n < r^n.$$

Therefore

$$d_I(0, v(x)) = \sup_{n \in \mathbb{N}} \sqrt{|R_n(x)|} < r.$$

Thus, $v(x) \in V(0, r)$. \hfill \Box
REFERENCES


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Вiдомо, що кожен комплекснозначний гомоморфiзм алгебри Фреше $H_{bs}(L_\infty)$ усiх цiлих симетричних функцiй обмеженого типу на комплексному банаховому просторi $L_\infty$ є функцiоналом обчислення значення в точцi $\delta_x$ (визначеного як $\delta_x(f) = f(x)$ для $f \in H_{bs}(L_\infty)$) у деякiй точцi $x \in L_\infty$. Тому спектр (множина усiх неперервних комплекснозначних гомоморфiзмiв) $M_{bs}$ алгебри $H_{bs}(L_\infty)$ є у взаємно однозначнiй вiдповiдi iз фактор-множиною $L_\infty/\sim$, де вiдношення еквiвалентностi "$\sim$" на просторi $L_\infty$ визначене наступним чином: $x \sim y \Longleftrightarrow \delta_x = \delta_y$. Як наслiдок, на $M_{bs}$ можна задати фактор-топологiю. З iншого боку, для $M_{bs}$ iснує природне по- дання у виглядi множини послiдовностей, яка разом iз заданими на нiй операцiєю покорординатного додавання i фактор-топологiєю утворює абелеву топологiчну групу. У статтi доведено, що топологiя на $M_{bs}$ є метризовною i породжується метрикою $d(\xi, \eta) = \sup_{n \in \mathbb{N}} \sqrt{\varepsilon_n - \eta_n}$, де $\xi = (\varepsilon_n)_{n=1}^\infty$, $\eta = (\eta_n)_{n=1}^\infty \in M_{bs}$.

Ключовi слова i фрази: симетрична функцiя, спектр алгебри.