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PROPERTIES OF POSITIVE CONTINUOUS FUNCTIONS IN $\mathbb{C}^n$

The properties of classes $Q^n_b$ and $Q$ of positive continuous functions are investigated. We prove that some compositions of functions from $Q$ belong to class $Q^n_b$. A relation between functions from these classes is established.

Key words and phrases: positive function, continuous function, several complex variables.

INTRODUCTION

Introducing entire functions of bounded $L$-index in direction (see [1]) we have to impose additional conditions to a continuous function $L : \mathbb{C}^n \to \mathbb{R}_+$. We suppose that $L \in Q^n_b$ (see below (5)). It is necessary to establish criteria of boundedness of $L$-index in direction and to apply $L$-index for solutions of partial differential equations or for entire functions with "plane" zeros [3].

Such conditions describe a behavior of slice function $L(z^0 + t\mathbf{b})$, $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$. It provides that function $L$ does not rapidly change as $|z| \to \infty$. In one-dimensional case Sheremeta M.M. [5] used a class $Q$ of positive continuous functions $l = l(t)$, $t \in \mathbb{C}$, satisfying some additional conditions. In fact, $l(t) = \ln |t|$, $l(t) = |t|^\alpha$, $\alpha \in \mathbb{R}_+$ belong to $Q$.

It is interesting: what are examples of functions from $Q^n_b$? To answer the question we consider compositions of functions from $Q$. Thus, it is a natural question: how to build a function $L \in Q^n_b$ by a function $l \in Q$?

1 PRELIMINARIES AND DENOTATIONS

For $\eta > 0$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{0\}$ and a positive continuous function $L : \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda^b_1(z, t_0, \eta) = \inf \left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \leq \frac{\eta}{L(z + t_0\mathbf{b})} \right\}, \quad (1)$$

$$\lambda^b_1(z, \eta) = \inf \{ \lambda^b_1(z, t_0, \eta) : t_0 \in \mathbb{C} \}, \quad (2)$$

and

$$\lambda^b_2(z, t_0, \eta) = \sup \left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \leq \frac{\eta}{L(z + t_0\mathbf{b})} \right\},$$

$$\lambda^b_2(\eta) = \inf \{ \lambda^b_2(z, \eta) : z \in \mathbb{C}^n \},$$

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\[ \lambda^b_2(z, \eta) = \sup \{ \lambda^b_2(z, t_0, \eta) : t_0 \in \mathbb{C} \}, \quad \lambda^b_1(\eta) = \sup \{ \lambda^b_1(z, \eta) : z \in \mathbb{C}^n \}. \] (4)

By \( Q^b \) we denote the class of functions \( L \), which for all \( \eta \geq 0 \) satisfy the condition
\[ 0 < \lambda^b_1(\eta) \leq \lambda^b_2(\eta) < +\infty. \] (5)

For a positive continuous function \( l(t) \) for \( t \in \mathbb{C} \) and \( t_0 \in \mathbb{C}, \) \( \eta > 0 \) we denote \( \lambda(\eta) = \lambda^b(0, t_0, \eta) \) and \( \lambda(\eta) = \lambda^b_2(0, t_0, \eta) \) in the case \( \eta = 0, b = 1, n = 1, L = l, \) and
\[ \lambda(\eta) = \inf \{ \lambda_1(t_0, \eta) : t_0 \in \mathbb{C} \}, \quad \lambda(\eta) = \sup \{ \lambda_2(t_0, \eta) : t_0 \in \mathbb{C} \}. \]

As in [5], by \( Q \) we denote the class of positive continuous functions \( l(t), t \in \mathbb{C}, \) which satisfy the condition: \( 0 < \lambda_1(\eta) \leq \lambda_2(\eta) < +\infty \) for all \( \eta \geq 0. \) In particular, \( Q = Q^1. \)

2 Elementary properties of functions from \( Q^b \)

Investigating the properties of entire functions of bounded \( L \)-index in direction we obtained following propositions about class \( Q^b. \)

Lemma 1 ([1]). If \( L \in Q^b_\theta \), then \( L \in Q^b_{\theta \eta} \) for every \( \theta \in \mathbb{C} \setminus \{0\} \), and if \( L \in Q^b_{\theta_1} \) and \( L \in Q^b_{\theta_2} \) then \( L \in Q^b_{\theta_1 + \theta_2} \) for any \( \theta_1, \theta_2 \in \mathbb{C}. \)

For \( l \in Q \) we denote
\[ l_1(t, w) = (|t| + |w| + 1)l(tw), \quad l_2(t, w) = (|w| + 1)l(tw), \quad l_3(t, w) = (|t| + 1)l(tw), \]
where \( t, w \in \mathbb{C}. \)

Lemma 2 ([2]). If \( l \in Q, \) then \( \forall \theta \in \mathbb{C} \) \( l_1 \in Q^2_\theta \), \( l_2 \in Q^2_\theta, \) \( l_3 \in Q^2_\theta, \) where \( \theta_1 = (1, 0), \) \( \theta_2 = (0, 1). \)

For \( l \in Q \) we denote \( l_4(z) = l(|z|), \) \( z \in \mathbb{C}. \)

Lemma 3 ([4]). If \( l \in Q, \) then \( l_4 \in Q^m \) for every \( m \in \mathbb{C} \) and every \( \theta \in \mathbb{C}. \)

For \( l \in Q \) we denote \( l_5(z) = l(|z|), \) \( z \in \mathbb{C}. \)

Lemma 4 ([4]). If \( l \in Q, \) then \( l_5 \in Q^b \) for every \( \theta \in \mathbb{C}. \)

It is easy to see that Lemmas 2, 3, 4 propose possible ways to construct a function \( L \in Q^b \) by a function \( l \in Q. \) Below we prove a generalization of Lemma 2 for \( \mathbb{C} \) (see Theorem 1).

Let \( L^* (z) \) be a positive continuous function in \( \mathbb{C}. \) The denotation \( L \prec L^* \) means that for some \( \theta_1, \theta_2 \in \mathbb{R}^+ \), and for all \( z \in \mathbb{C} \) the inequalities \( \theta_1 L^* (z) \leq \theta_2 L(z) \) hold.

Lemma 5. If \( L \in Q^b, \) \( L \prec L^* \), then \( L^* \in Q^b. \)

Proof. Using the definition of \( Q^b, \) we have
\[
\inf_{z \in \mathbb{C}} \inf_{t_0 \in \mathbb{C}} \inf \left\{ \frac{L^*(z + t_0 \theta b)}{L^*(z + t_0 b)} : |t - t_0| \leq \frac{\eta}{L^*(z + t_0 b)} \right\} \\
\geq \inf_{z \in \mathbb{C}} \inf_{t_0 \in \mathbb{C}} \inf \left\{ \frac{\theta_1 L(z + t_0 \theta b)}{\theta_2 L(z + t_0 b)} : |t - t_0| \leq \frac{\eta}{\theta_1 L(z + t_0 b)} \right\} \\
= \frac{\theta_1}{\theta_2} \inf_{z \in \mathbb{C}} \inf_{t_0 \in \mathbb{C}} \inf \left\{ \frac{L(z + t_0 \theta b)}{L(z + t_0 b)} : |t - t_0| \leq \frac{\eta}{\theta_1 L(z + t_0 b)} \right\} > 0,
\]
because \( L \in Q^n_b \). Besides,

\[
\sup_{z \in \mathbb{C}^n} \inf_{t_0 \in \mathbb{C}} \left\{ \frac{L^*(z + t_0b)}{L^*(z + t_0b)} : |t - t_0| \leq \frac{\eta}{L^*(z + t_0b)} \right\} 
\leq \sup_{z \in \mathbb{C}^n} \inf_{t_0 \in \mathbb{C}} \left\{ \frac{\theta_2 L(z + t_0b)}{\theta_1 L(z + t_0b)} : |t - t_0| \leq \frac{\eta}{\theta_1 L(z + t_0b)} \right\} 
= \frac{\theta_2}{\theta_1} \sup_{z \in \mathbb{C}^n} \inf_{t_0 \in \mathbb{C}} \left\{ \frac{L(z + t_0b)}{L(z + t_0b)} : |t - t_0| \leq \frac{\eta}{\theta_1 L(z + t_0b)} \right\} < +\infty.
\]

Thus \( L^* \in Q^n_b \). □

3 Main theorem

Now we prove several propositions that indicate ways of construction of functions from the class \( Q^n_b \).

**Theorem 1.** If \( l \in Q \) and \( \inf \{l(t) : t \in \mathbb{C} \} = c > 0 \), then \( L \in Q^n_b \), where

\[
L(z) = \frac{1}{c} \left( 1 + \sum_{k=1}^n \left( |b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^n (|z_j| + |b_j|) \right) l \left( \prod_{j=1}^n z_j \right) \right)
\]

and \( \prod_{j \in \mathbb{C}} (\cdot) = 1 \).

**Proof.** Note that in the definition of \( Q^n_b \) it is required that inequality (5) holds for all \( \eta > 0 \). But in view of (1)–(4) function \( \lambda^b_1(\eta) \) is nonincreasing and \( \lambda^b_2(\eta) \) is nondecreasing. So it is sufficient to require in definition of \( Q^n_b \) that inequality (5) is true for all \( \eta \geq 1 \). Indeed let this inequality holds for \( \eta^* > 1 \). Then for all \( \eta \) such that \( 0 < \eta < \eta^* \leq +\infty \), the following inequalities hold \( \lambda^b_1(\eta) \geq \lambda^b_1(\eta^*) > 0, \lambda^b_2(\eta) \leq \lambda^b_2(\eta^*) < +\infty \). Thus inequality (5) holds for all \( \eta > 0 \). Below we assume that \( \eta \geq 1 \).

Besides, we suppose that \( \inf \{l(t) : t \in \mathbb{C} \} = 1 \). If this infimum does not equal 1, then we can consider the function \( \tilde{l}(t) = \frac{l(t)}{\inf \{l(t) : t \in \mathbb{C} \}} \), for which this equality holds.

So we consider the case \( \eta \geq 1 \) and \( \inf \{l(t) : t \in \mathbb{C} \} = 1 \). We shall prove that for all \( \eta \geq 1 \) the following inequalities hold

\[
\inf_{z \in \mathbb{C}^n} \inf_t \left\{ \left( 1 + \sum_{k=1}^n \left( |b_k| \prod_{j=1}^{k-1} |z_j| + b_j t \prod_{j=k+1}^n (|z_j| + |b_j|) \right) l \left( \prod_{j=1}^n (z_j + b_j t) \right) \right) \right\} \leq \frac{\eta}{\left( 1 + \sum_{k=1}^n \left( |b_k| \prod_{j=1}^{k-1} |z_j| + b_j t^0 \prod_{j=k+1}^n (|z_j| + b_j t^0) \right) l \left( \prod_{j=1}^n (z_j + b_j t^0) \right) \right)} \cdot \frac{1}{\prod_{j \in \mathbb{C}} (z_j + b_j t^0)} > 0
\]

(6)
We estimate each of obtained $n$ differences separately. In particular $n$-th difference can be estimated as

$$
\left| \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0) \right| = \left| \left( \prod_{j=1}^{n}(z_j + b_j t) - (z_1 + b_1 t^0) \prod_{j=2}^{n}(z_j + b_j t) \right) + \cdots + \left( z_j + b_j t^0 \right) \prod_{j=1}^{n}(z_j + b_j t) - \left( \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{k}(z_j + b_j t^0) \right) \prod_{j=k+1}^{n}(z_j + b_j t) \right| + \cdots
$$

We estimate each of obtained $n$ differences separately. In particular $n$-th difference can be estimated as

$$
\left| \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0) \right| = \left| \prod_{j=1}^{n}(z_j + b_j t) - (z_1 + b_1 t^0) \prod_{j=2}^{n}(z_j + b_j t) \right| + \cdots + \left( z_j + b_j t^0 \right) \prod_{j=1}^{n}(z_j + b_j t) - \left( \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{k}(z_j + b_j t^0) \right) \prod_{j=k+1}^{n}(z_j + b_j t) \right| + \cdots
$$

Hence, we obtain that

$$
|t - t^0| \leq \frac{\eta}{\left( 1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{n} |z_j + b_j t^0| \prod_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j|) \right) \right) 1 \left( \prod_{j=1}^{n}(z_j + b_j t^0) \right)} \leq \eta.
$$

It remains to estimate the module

$$
\left| \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0) \right| = \left| \left( \prod_{j=1}^{n}(z_j + b_j t) - (z_1 + b_1 t^0) \prod_{j=2}^{n}(z_j + b_j t) \right) + \cdots + \left( z_j + b_j t^0 \right) \prod_{j=1}^{n}(z_j + b_j t) - \left( \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{k}(z_j + b_j t^0) \right) \prod_{j=k+1}^{n}(z_j + b_j t) \right| + \cdots
$$

We estimate each of obtained $n$ differences separately. In particular $n$-th difference can be estimated as

$$
\left| \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0) \right| = \prod_{j=1}^{n-1} |z_j + b_j t_0| |b_n|.
$$

Applying the inequality (8) and using that $\eta > 1$, $(n - 1)$-th differences can be estimated as

$$
\left| \prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0) \right| = \left( \prod_{j=1}^{n-1} |z_j + b_j t_0| |b_n| \right) \left( \prod_{j=1}^{n}(z_j + b_j t^0) \right).
$$
\[
\left| \prod_{j=1}^{n-2} (z_j + b_j t^0) \prod_{j=n-1}^{n} (z_j + b_j t) - \prod_{j=1}^{n-1} (z_j + b_j t^0) (z_j + b_j t) \right| = \prod_{j=1}^{n-2} |z_j + b_j t^0| |b_{n-1}| |t - t^0| |z_n + b_n t|
\]
\[
= \prod_{j=1}^{n-2} |z_j + b_j t^0| |b_{n-1}| |t - t^0| \left| z_n + b_n t^0 + b_n (t - t^0) \right|
\]
\[
\leq \prod_{j=1}^{n-2} |z_j + b_j t^0| |b_{n-1}| |t - t^0| + \prod_{j=1}^{n-2} |z_j + b_j t^0| |b_{n-1}| |b_n| |t - t^0|^2
\]
\[
\leq \eta \prod_{j=1, j \neq n-1}^{n} |z_j + b_j t_0| |b_{n-1}|
\]
\[
\leq \frac{\eta^2 \prod_{j=1, j \neq n-1}^{n} |z_j + b_j t_0| |b_{n-1}| + n \prod_{j=1}^{n} (|z_j + b_j t_0| + |b_j|)}{\left( 1 + \sum_{j=1}^{n} (|b_j| |z_j + b_j t_0| + |b_j|) \right) l \left( \prod_{j=1}^{n} |z_j + b_j t_0| \right)}
\]
\[
= \frac{\eta^2 \prod_{j=1, j \neq n-1}^{n} |z_j + b_j t_0| |b_{n-1}| + n \prod_{j=1}^{n} (|z_j + b_j t_0| + |b_j|)}{\left( 1 + \sum_{j=1}^{n} (|b_j| |z_j + b_j t_0| + |b_j|) \right) l \left( \prod_{j=1}^{n} |z_j + b_j t_0| \right)} \cdot \frac{1}{\left( 1 + \prod_{j=1}^{n} (|z_j + b_j t_0| + |b_j|) \right) l \left( \prod_{j=1}^{n} (z_j + b_j t_0) \right)}
\]

For arbitrary \( k \)-th difference, \( 1 \leq k \leq n \), of (9) we can obtain estimate
\[
\left| \prod_{j=1}^{k-1} (z_j + b_j t^0) \prod_{j=k}^{n} (z_j + b_j t) - \prod_{j=1}^{k} (z_j + b_j t^0) \prod_{j=k+1}^{n} (z_j + b_j t) \right|
\]
\[
= \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} |z_j + b_j t| |b_k| |t - t_0|
\]
\[
= \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} |z_j + b_j t^0 + b_j (t - t^0)| |b_k| |t - t_0|
\]
\[
\leq \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j| |t - t^0|) |b_k| |t - t_0|
\]
\[
\leq \eta |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j| |t - t^0|) l \left( \prod_{j=1}^{n} (z_j + b_j t_0) \right)
\]
\[
\leq \eta^{n-k} |b_k| \prod_{j=1}^{n} (|z_j + b_j t^0| + |b_j|) \prod_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j|) l \left( \prod_{j=1}^{n} (z_j + b_j t_0) \right)
\]

Thus, returning to (9) and considering that \( \eta^j \leq \eta^n \) for all \( j, 1 \leq j \leq n \), we obtain the following inequality.
\[
\left| \prod_{j=1}^{n} (z_j + b_j t) - \prod_{j=1}^{n} (z_j + b_j t^0) \right| \\
\leq \sum_{k=1}^{n} \frac{\eta^{n-k} |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} \left( |z_j + b_j t^0| + |b_j| \right)}{1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} \left( |z_j + b_j t^0| + |b_j| \right) \right)} \frac{1}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \\
\leq \eta \sum_{k=1}^{n} \frac{|b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} \left( |z_j + b_j t^0| + |b_j| \right)}{1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} \left( |z_j + b_j t^0| + |b_j| \right) \right)} \frac{1}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \\
\times \frac{1}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \leq \frac{1}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)}
\]

Then for all \( \eta \geq 1 \)

\[
\inf_{z \in \mathbb{C}^n} \inf_{\mu \in \mathcal{C}} \inf_{t} \left\{ \frac{1}{\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \frac{1}{1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} \left( |z_j + b_j t^0| + |b_j| \right) \right)} \frac{1}{l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)} : |t - t^0| \leq \eta \right\}
\]

\[
\geq \inf_{z \in \mathbb{C}^n} \inf_{\mu \in \mathcal{C}} \inf_{t} \left\{ \frac{1}{\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \frac{1}{1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} \left( |z_j + b_j t^0| + |b_j| \right) \right)} \frac{1}{l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)} : |t - t^0| \leq \eta \right\}
\]

\[
\times \inf_{z \in \mathbb{C}^n} \inf_{\mu \in \mathcal{C}} \inf_{t} \left\{ \frac{l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} : \left| \prod_{j=1}^{n} (z_j + b_j t) - \prod_{j=1}^{n} (z_j + b_j t^0) \right| \leq \frac{l}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right\}.
\]

(10)

The first factor in the obtained inequality is a fractional rational expression with the same
degrees of the numerator and denominator by variable \( z_j \), and by \( t, t^0 \), respectively. Thus the corresponding infimum is not equal to zero. Suppose that the second expression equals zero.

Then there exists sequences \( (z^p), (t^p) \), for which

\[
\inf_t \left\{ \frac{l\left( \prod_{j=1}^{n} (z^p_j + b_j t) \right)}{l\left( \prod_{j=1}^{n} (z^p_j + b_j t^0) \right)} : \left| \prod_{j=1}^{n} (z^p_j + b_j t) - \prod_{j=1}^{n} (z^p_j + b_j t^0) \right| \leq \frac{\eta^n}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right\} \to 0.
\]

Denoting \( u_p(t) = \prod_{j=1}^{n} (z^p_j + b_j t) \), and \( v_p(t^0) = \prod_{j=1}^{n} (z^p_j + b_j t^0) \), we obtain that

\[
\inf_t \left\{ \frac{l(u_p(t))}{l(v_p(t^0))} : |u_p(t) - v_p(t^0)| \leq \frac{\eta}{l(v_p(t^0))} \right\} \to 0.
\]

But

\[
\inf_{t} \left\{ \frac{l(u_p(t))}{l(v_p(t^0))} : |u_p(t) - v_p(t^0)| \leq \frac{\eta}{l(v_p(t^0))} \right\} \geq \inf_{u,v} \left\{ \frac{l(u)}{l(v)} : |u - v| \leq \frac{\eta}{l(v)} \right\},
\]

and \( \inf_{u,v} \left\{ \frac{l(u)}{l(v)} : |u - v| \leq \frac{\eta}{l(v)} \right\} = 0 \), that contradicts the condition \( l \in Q \). Thus, the second factor in (10) is also positive, so the inequality (6) is correct.

Using similar considerations, we can prove the similar inequality for \( \sup \). Indeed, for all \( \eta \geq 1 \) the following inequalities hold

\[
\sup_{z \in \mathbb{C}^n, r \in \mathbb{C}} \sup_{t} \left\{ \frac{1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + b_j t| \prod_{j=k+1}^{n} (|z_j + b_j t| + |b_j|) \right)}{1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j|) \right)} \right\} \left( \frac{\eta}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right) \leq \sup_{z \in \mathbb{C}^n, r \in \mathbb{C}} \sup_{t} \left\{ \frac{n}{l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)} : \left| \prod_{j=1}^{n} (z_j + b_j t) - \prod_{j=1}^{n} (z_j + b_j t^0) \right| \leq \frac{\eta^n}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right\},
\]

\[
|t - t^0| \leq \sup_{z \in \mathbb{C}^n, r \in \mathbb{C}} \sup_{t} \left\{ \frac{1}{l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)} : \left| \prod_{j=1}^{n} (z_j + b_j t) - \prod_{j=1}^{n} (z_j + b_j t^0) \right| \leq \frac{\eta^n}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right\}.
\]

(11)
\[ \leq \sup_{z \in \mathbb{C}^n} \sup_{\rho \in \mathbb{C}} \sup_t \left\{ \left( 1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + |b_j| t_j | + |b_j| \right) \right) \prod_{j=k+1}^{n} (|z_j + b_j t_j | + |b_j|)) \right\} : |t - t^0| \leq \eta \right\}. \]

As above for infimum in the first brackets we obtain a fractional rational expression with the same degrees of the numerator and denominator by \( z_j \), and by \( t \), \( t^0 \) respectively. Hence corresponding supremum does not equal infinity. Suppose that the second expression is equal to infinity. Then there exist \((z^p), (t^0_p)\) with property

\[ \sup_{t} \left\{ \frac{l\left( \prod_{j=1}^{n} (z_j^p + b_j t_j) \right)}{l\left( \prod_{j=1}^{n} (z_j^p + b_j t^0_p) \right)} : \left| \prod_{j=1}^{n} (z_j^p + b_j t_j) - \prod_{j=1}^{n} (z_j^p + b_j t^0_p) \right| \leq \frac{\eta^n}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0_p) \right)} \right\} \rightarrow_{p \rightarrow \infty} \infty. \]

Denoting \( u_p(t) = \prod_{j=1}^{n} (z_j^p + b_j t_j) \), and \( v_p(t^0_p) = \prod_{j=1}^{n} (z_j^p + b_j t^0_p) \), we obtain

\[ \sup_{t} \left\{ \frac{l(u_p(t))}{l(v_p(t^0_p))} : |u_p(t) - v_p(t^0_p)| \leq \frac{\eta}{l(v_p(t^0_p))} \right\} \rightarrow_{p \rightarrow \infty} \infty, \]

But

\[ \sup_{t} \left\{ \frac{l(u_p(t))}{l(v_p(t^0_p))} : |u_p(t) - v_p(t^0_p)| \leq \frac{\eta}{l(v_p(t^0_p))} \right\} \leq \sup_u \left\{ \frac{l(u)}{l(v(t^0))} : |u - v(t^0)| \leq \frac{\eta}{l(v(t^0))} \right\}, \]

and \( \sup_{\nu \in \mathbb{C}} \sup_u \left\{ \frac{l(u)}{l(v)} : |u - v| \leq \frac{\eta}{l(v)} \right\} = \infty \), that contradicts the condition \( l \in \mathbb{Q} \). Thus, the second factor in (11) is also positive, so the inequality (7) is valid. Hence, we deduce that the function

\[ \frac{1}{c} \left( 1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^{n} (|z_j| + |b_j|) \right) \right) l\left( \prod_{j=1}^{n} z_j \right) \]

belongs to the class \( \mathbb{Q}_b^u \).

\[ \square \]

4 REMARKS TO MAIN THEOREM

\textbf{Remark 1.} The condition \( \inf\{l(t) : t \in \mathbb{C} \} = c > 0 \) is not essential. In fact, every function \( l \in \mathbb{Q} \), which satisfies the equality \( \inf\{l(t) : t \in \mathbb{C} \} = 0 \), can be replaced by the function \( l(t) + 1 \), which also belongs to the class \( \mathbb{Q} \).

\textbf{Proof.} Indeed, for the positive continuous function \( l(t) \) the inequality holds

\[ \frac{l(t)}{l(t_0)} \leq \frac{l(t) + 1}{l(t_0) + 1} < \frac{l(t)}{l(t_0)} + 1, \]

where the right part is true for all \( t, t_0 \in \mathbb{C} \), and the left part is true for all \( t, t_0 \in \mathbb{C} \) such that \( l(t) \leq l(t_0) \). The right inequality is equivalent to the following

\[ l(t_0)(l(t) + 1) < (l(t) + l(t_0))(l(t_0) + 1) \quad \text{or} \quad l(t_0)(l(t) + l(t_0) < l(t)l(t_0) + l^2(t_0) + l(t) + l(t_0),} \]
i.e. $0 < l^2(t_0) + l(t)$. But this inequality holds for the function $l(t)$ for all $t, t_0 \in \mathbb{C}$.

From the left part we similarly obtain $l(t)l(t_0) + l(t) \leq l(t_0)(l(t) + 1)$. Hence $l(t) \leq l(t_0)$.

Evaluating the supremum for the right part of inequality (12) and the infimum for the left side and using that $l(t) \in Q$, we obtain

$$0 < \inf \left\{ \frac{l(t)}{l(t_0)} : |t - t_0| \leq \frac{\eta}{l(t_0)}, t \in \mathbb{C} \right\} \leq \inf \left\{ \frac{l(t)}{l(t_0)} : |t - t_0| \leq \frac{\eta}{l(t_0) + 1}, t \in \mathbb{C} \right\}$$

$$\leq \inf \left\{ \frac{l(t) + 1}{l(t_0) + 1} : |t - t_0| \leq \frac{\eta}{l(t_0) + 1}, t \in \mathbb{C} \right\}$$

$$\leq \sup \left\{ \frac{l(t)}{l(t_0)} + 1 : |t - t_0| \leq \frac{\eta}{l(t_0) + 1}, t \in \mathbb{C} \right\} \leq \sup \left\{ \frac{l(t) + 1}{l(t_0) + 1} : |t - t_0| \leq \frac{\eta}{l(t_0) + 1}, t \in \mathbb{C} \right\} \leq \sup \left\{ \frac{l(t)}{l(t_0)} + 1 : |t - t_0| \leq \frac{\eta}{l(t_0)}, t \in \mathbb{C} \right\} < \infty.$$

These inequalities imply $l(t) + 1 \in Q$.

**Remark 2.** In fact, analysis of the proof of Theorem 1 indicates that we can somehow decrease function $L$. For each $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$, such that $\prod_{j=1}^n |b_j| \neq 0$, $l \in Q$ and $\inf \{l(t) : t \in \mathbb{C} \} = c > 0$, we have $L \in Q^n_b$, where

$$L(z) = \frac{1}{c} \left( \sum_{k=1}^n \prod_{j=1}^{k-1} |b_j| \prod_{j=k+1}^n (|z_j| + |b_j|) \right).$$

The appearance of term 1 in the proof of Theorem 1 is necessary for lower estimate of the function $\left( \sum_{k=1}^n \left( |b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^n (|z_j| + |b_j|) \right) \right)^j$, where $j = 1, 2, \ldots, n$. We can take the direction $\tilde{b} = b / \prod_{j=1}^n |b_j|$ instead of $b$ under the previous condition $\prod_{j=1}^n |b_j| \neq 0$, because by Lemma 1 the function $L$ belongs to the class $Q^n_{\tilde{b}b}$, with $\theta = \frac{1}{\prod_{j=1}^n |b_j|}$.

Then all considerations of previous theorem should be repeated, omitting the term 1 in the appropriate places. Alternatively we can take a larger function.

**Remark 3.** If $l^* \in Q$, $l \in Q$, $\inf \{l(t) : t \in \mathbb{C} \} = c > 0$, and for all $z \in \mathbb{C}^n$ the following inequalities hold

$$l^* \left( \prod_{j=1}^n z_j \right) \geq c_1 \left( 1 + \sum_{k=1}^n \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^n (|z_j| + |b_j|) \right)$$

and

$$l^* \left( \prod_{j=1}^n z_j \right) \leq c_2 \left( \prod_{j=1}^n (|z_j| + |b_j|) - \prod_{j=1}^n |z_j| \right),$$

then $L \in Q^n_b$, where $L(z) = \frac{1}{c} l^* \left( \prod_{j=1}^n z_j \right) l^{n} \left( \prod_{j=1}^n z_j \right)$, $b = (b_1, b_2, \ldots, b_n)$, $c_1 > 0$, $c_2 > 0$. 
Proof. Without loss of generality, we may suppose $\inf \{l(t) : t \in \mathbb{C} \} = 1$ as in Theorem 1. Then we can repeat the considerations of this theorem, taking everywhere the function $l^*\left( \prod_{j=1}^{n} z_j \right)$ instead of

$$1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^{n} (|z_j| + |b_j|) \right).$$

Therefore we obtain

$$\left| \prod_{j=1}^{n} (z_j + b_j t) - \prod_{j=1}^{n} (z_j + b_j t^0) \right| \leq \frac{\sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j + b_j t^0| \prod_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j|) \right)}{\min \{1, c^n \} l^*\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \frac{l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)}.$$ 

$$\leq \frac{\eta^n}{\min \{c_1, c^n+1 \} l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)}.$$ 

Denoting $\tilde{c} = \min \{c_1, c_1^{n+1} \}$, for all $\eta \geq 1$ we obtain the following inequality

$$\inf_{z \in \mathbb{C}^n} \inf_{\rho \in \mathbb{C}} \inf_{t} \left\{ \frac{l^*\left( \prod_{j=1}^{n} (z_j + b_j t) \right) l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)}{l^*\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right) l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} : \left| \prod_{j=1}^{n} (z_j + b_j t) - \prod_{j=1}^{n} (z_j + b_j t^0) \right| \leq \frac{\eta^n}{\tilde{c} l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right\}.$$ 

$$\times \inf_{z \in \mathbb{C}^n} \inf_{\rho \in \mathbb{C}} \inf_{t} \left\{ \frac{l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)}{l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} : \left| \prod_{j=1}^{n} (z_j + b_j t) - \prod_{j=1}^{n} (z_j + b_j t^0) \right| \leq \frac{\eta^n}{\tilde{c} l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right\}.$$ 

(13)

Since $l(t) \in \mathcal{Q}$, by similar considerations as in Theorem 1 it can be showed that the product in (13) is greater than zero. It is obviously that we can prove

$$\sup_{z \in \mathbb{C}^n} \sup_{\rho \in \mathbb{C}} \sup_{t} \left\{ \frac{l^*\left( \prod_{j=1}^{n} (z_j + b_j t) \right) l\left( \prod_{j=1}^{n} (z_j + b_j t) \right)}{l^*\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right) l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} : \left| t - t^0 \right| \leq \frac{\eta}{l^*\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right) l\left( \prod_{j=1}^{n} (z_j + b_j t^0) \right)} \right\} < \infty.$$ 

(14)
In view of (13), (14) we obtain that the function $l^*\left(\prod_{j=1}^{n}|z_j|\right)l\left(\prod_{j=1}^{n}|z_j|\right)$ belongs to the class $Q^n_b$.

**Remark 4.** We can take the following functions

$$1 + \prod_{j=1}^{n}(|z_j| + |b_j|) - \prod_{j=1}^{n} |z_j| \quad \text{or} \quad \sum_{k=1}^{n} \left( \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^{n} (|z_j| + |b_j|) \right)$$

instead of the expression

$$\prod_{j=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^{n} (|z_j| + |b_j|) \right)$$

in Theorem 1.

It follows from Lemma 5 and notion

$$1 + \prod_{j=1}^{n}(|z_j| + |b_j|) - \prod_{j=1}^{n} |z_j| \sim 1 + \sum_{k=1}^{n} \left( |b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^{n} (|z_j| + |b_j|) \right).$$

**Proposition 1.** If $L \in Q^n_b$, then for every $z^0 \in \mathbb{C}^n$ we have $l_{z^0} \in Q \left( l_{z^0}(t) \equiv L(z^0 + t|b|) \right)$.

**Proof.** We remark that (1)–(5) imply for every $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$

$$\forall \eta > 0 \ 0 < \lambda_1^b(z, \eta) \leq \lambda_1^b(z, t_0, \eta) \leq 1 \leq \lambda_2^b(z, t_0, \eta) \leq \lambda_2^b(z, \eta) < +\infty.$$  

These inequalities imply that $l_{z^0} \in Q$.

**References**


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Досліджено властивості класів $Q^n_b$ та $Q$ додатних неперервних функцій. Доведено, що деякі композиції функцій із класу $Q$ належать класу $Q^n_b$. Встановлено зв’язок між функціями цих класів.

Ключові слова і фрази: додатна функція, неперервна функція, декілька комплексних змінних.