CHARACTERIZATIONS OF REGULAR AND INTRA-REGULAR ORDERED \( \Gamma \)-SEMIHYPERGROUPS IN TERMS OF BI-\( \Gamma \)-HYPERIDEALS

The concept of \( \Gamma \)-semihypergroups is a generalization of semigroups, a generalization of semihypergroups and a generalization of \( \Gamma \)-semigroups. In this paper, we study the notion of bi-\( \Gamma \)-hyperideals in ordered \( \Gamma \)-semihypergroups and investigate some properties of these bi-\( \Gamma \)-hyperideals. Also, we define and use the notion of regular ordered \( \Gamma \)-semihypergroups to examine some classical results and properties in ordered \( \Gamma \)-semihypergroups.

Key words and phrases: ordered \( \Gamma \)-semihypergroup, \( \Gamma \)-hyperideal, bi-\( \Gamma \)-hyperideal.

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1 Introduction

A semigroup is an algebraic structure consisting of a non-empty set \( S \) together with an associative binary operation [24]. The notion of a \( \Gamma \)-semigroup was introduced by Sen and Saha [37] as a generalization of semigroups as well as of ternary semigroups. Since then, hundreds of papers have been written on this topic, see [6, 7, 16]. Many classical notions of semigroups have been extended to \( \Gamma \)-semigroups. Let \( S = \{a, b, c, \ldots \} \) and \( \Gamma = \{\alpha, \beta, \gamma, \ldots \} \) be two non-empty sets. Then, \( S \) is called a \( \Gamma \)-semigroup if there exists a mapping from \( S \times \Gamma \times S \) to \( S \), written as \((a, \gamma, b) \rightarrow a\gamma b\), satisfying the identity \((a\alpha b)\beta c = a(a\beta b)c\) for all \(a, b, c \in S\) and \(\alpha, \beta \in \Gamma\). In this case by \((S, \Gamma)\) we mean \( S \) is a \( \Gamma \)-semigroup. By an ordered semigroup, we mean an algebraic structure \((S, \cdot, \leq)\), which satisfies the following conditions: (1) \((S, \cdot)\) is a semigroup; (2) \( S \) is a partial ordered set by \(\leq\); (3) If \(a\) and \(b\) are elements of \( S \) such that \(a \leq b\), then \(a \cdot c \leq b \cdot c\) and \(c \cdot a \leq c \cdot b\) for all \(c \in S\). Ordered semigroups have been studied extensively by Kehayopulu and Tsingelis, for example, see [27–29]. The notions of an ordered \( \Gamma \)-groupoid and an ordered \( \Gamma \)-semigroup were defined by Sen and Seth in [38]. Many authors studied different aspects of ordered \( \Gamma \)-semigroups, for instance, Abbasi and Basar [1], Chinram and Tinpun [7, 8], Dutta and Adhikari [16, 17], Hila [22], Iampan [25], Kehayopulu [26], Kwon [31], Kwon and Lee [32, 33], and many others. Recall from [38], that an ordered \( \Gamma \)-semigroup \((S, \Gamma, \leq)\) is a \( \Gamma \)-semigroup \((S, \Gamma)\) together with an order relation \(\leq\) such that \(a \leq b\) implies that \(a\gamma c \leq b\gamma c\) and \(c\gamma a \leq c\gamma b\) for all \(a, b, c \in S\) and \(\gamma \in \Gamma\).

The concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. The concept of ordering hypergroups introduced by Chvalina [11] as a special class
of hypergroups. Many authors studied different aspects of ordered semihypergroups, for instance, Davvaz et al. [15], Gu and Tang [19], Heidari and Davvaz [20], Tang et al. [39], and many others. Explicit study of ordered semihypergroups seems to have begun with Heidari and Davvaz [20] in 2011. Recall from [20], that an ordered semihypergroup \((S, \circ, \leq)\) is a semihypergroup \((S, \circ)\) together with a partial order \(\leq\) that is compatible with the hyperoperation \(\circ\), meaning that for any \(x, y, z \in S\),

\[ x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z. \]

Here, \(z \circ x \leq z \circ y\) means for any \(a \in z \circ x\) there exists \(b \in z \circ y\) such that \(a \leq b\). The case \(x \circ z \leq y \circ z\) is defined similarly.

Recently, Davvaz et al. [4, 5, 13, 21, 23] studied the notion of \(\Gamma\)-semihypergroups as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a \(\Gamma\)-semigroup. They proved some results in this respect and presented many examples of \(\Gamma\)-semihypergroups. Many classical notions of semigroups and semihypergroups have been extended to \(\Gamma\)-semihypergroups. The notion of a \(\Gamma\)-hyperideal of a \(\Gamma\)-semihypergroup was introduced in [4]. Davvaz et al. [5] introduced the notion of Pawlak’s approximations in \(\Gamma\)-semihypergroups. Abdullah et al. [2] studied \(-\)hypersystems and \(N\)-hypersystems in a \(\Gamma\)-semihypergroup. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The concept of hyperstructure was first introduced by Marty [34] at the eighth Congress of Scandinavian Mathematicians in 1934. A comprehensive review of the theory of hyperstructures can be found in [9, 10, 12, 40]. Let \(S\) be a non-empty set and \(P^*(S)\) be the family of all non-empty subsets of \(S\). A mapping \(\circ : S \times S \rightarrow P^*(S)\) is called a hyperoperation on \(S\). A hypergroupoid is a set \(S\) together with a (binary) hyperoperation. In the above definition, if \(A\) and \(B\) are two non-empty subsets of \(S\) and \(x \in S\), then we denote

\[ A \circ B = \bigcup_{x \in A} x \circ A = \{x\} \circ A \quad \text{and} \quad B \circ x = B \circ \{x\}. \]

A hypergroupoid \((S, \circ)\) is called a semihypergroup if for every \(x, y, z \in S, x \circ (y \circ z) = (x \circ y) \circ z\). That is,

\[ \bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z. \]

A non-empty subset \(K\) of a semihypergroup \(S\) is called a subsemihypergroup of \(S\) if \(K \circ K \subseteq K\). A hypergroupoid \((S, \circ)\) is called a quasihypergroup if for every \(x \in S, x \circ S = S = S \circ x\). This condition is called the reproduction axiom. The couple \((S, \circ)\) is called a hypergroup if it is a semihypergroup and a quasihypergroup. A non-empty subset \(K\) of \(S\) is a subhypergroup of \(S\) if \(K \circ a = a \circ K = K\), for every \(a \in K\). A hypergroup \((S, \circ)\) is called commutative if \(x \circ y = y \circ x\), for every \(x, y \in S\).

2 Review: Ordered \(\Gamma\)-Semihypergroups

The notion of a \(\Gamma\)-semihypergroup was introduced by Davvaz et al. [4, 5, 21]. In [20], Heidari and Davvaz introduced the concept of ordered semihypergroups, which is a generalization of
ordered semigroups. In this section, we recall the notion of an ordered \( \Gamma \)-semihypergroup and then we present some definitions and properties which we will need in this paper. Throughout this paper, unless otherwise stated, \( S \) is always an ordered \( \Gamma \)-semihypergroup \((S, \Gamma, \leq)\).

**Definition 1** ([4, 5]). Let \( S \) and \( \Gamma \) be two non-empty sets. Then, \( S \) is called a \( \Gamma \)-semihypergroup if every \( \gamma \in \Gamma \) is a hyperoperation on \( S \), i.e., \( x\gamma y \subseteq S \) for every \( x, y \in S \), and for every \( \alpha, \beta \in \Gamma \) and \( x, y, z \in S \), we have \( x\alpha(y\beta z) = (x\alpha y)\beta z \). If every \( \gamma \in \Gamma \) is an operation, then \( S \) is a \( \Gamma \)-semigroup. Let \( A \) and \( B \) be two non-empty subsets of \( S \). We define

\[
A \Gamma B = \bigcup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\} = \bigcup_{\gamma \in \Gamma} A \gamma B.
\]

A \( \Gamma \)-semihypergroup \( S \) is called commutative if for all \( x, y \in S \) and \( \gamma \in \Gamma \), we have \( x\gamma y = y\gamma x \). A \( \Gamma \)-semihypergroup \( S \) is called a \( \Gamma \)-hypergroup if for every \( \gamma \in \Gamma \), \( (S, \gamma) \) is a hypergroup.

Now, we consider the notion of an ordered \( \Gamma \)-semihypergroup.

**Definition 2** ([30]). An algebraic hyperstructure \((S, \Gamma, \leq)\) is called an ordered \( \Gamma \)-semihypergroup if \((S, \Gamma)\) is a \( \Gamma \)-semihypergroup and \((S, \leq)\) is a partially ordered set such that for any \( x, y, z \in S \) and \( \gamma \in \Gamma \), \( x \leq y \) implies \( z\gamma x \leq z\gamma y \) and \( x\gamma z \leq y\gamma z \). Here, if \( A \) and \( B \) are two non-empty subsets of \( S \), then we say that \( A \leq B \) if for every \( a \in A \) there exists \( b \in B \) such that \( a \leq b \).

Let \( S \) be an ordered \( \Gamma \)-semihypergroup. By a sub \( \Gamma \)-semihypergroup of \( S \) we mean a non-empty subset \( A \) of \( S \) such that \( a\gamma b \subseteq A \) for all \( a, b \in A \) and \( \gamma \in \Gamma \).

**Example 1** ([30]). Let \((S, \circ, \leq)\) be an ordered semihypergroup and \( \Gamma \) a non-empty set. We define \( x\gamma y = x \circ y \) for every \( x, y \in S \) and \( \gamma \in \Gamma \). Then, \((S, \Gamma, \leq)\) is an ordered \( \Gamma \)-semihypergroup.

**Definition 3.** Let \((S, \Gamma, \leq)\) be an ordered \( \Gamma \)-semihypergroup. A non-empty subset \( I \) of \( S \) is called a left \( \Gamma \)-hyperideal of \( S \) if it satisfies the following conditions:

1. \( S \Gamma I \subseteq I \);
2. When \( x \in I \) and \( y \in S \) such that \( y \leq x \), imply that \( y \in I \).

A right \( \Gamma \)-hyperideal of an ordered \( \Gamma \)-semihypergroup \( S \) is defined in a similar way. By two-sided \( \Gamma \)-hyperideal or simply \( \Gamma \)-hyperideal, we mean a non-empty subset of \( S \) which both left and right \( \Gamma \)-hyperideal of \( S \). A \( \Gamma \)-hyperideal \( I \) of \( S \) is said to be proper if \( I \neq S \).

Let \( K \) be a non-empty subset of an ordered \( \Gamma \)-semihypergroup \((S, \Gamma, \leq)\). If \( H \) is a non-empty subset of \( K \), then we define \((H)_K := \{k \in K \mid k \leq h \text{ for some } h \in H\}\). Note that if \( K = S \), then we define \((H) := \{x \in S \mid x \leq h \text{ for some } h \in H\}\). For \( H = \{h\} \), we write \( (h) \) instead of \((\{h\})\). Note that the condition (2) in Definition 3 is equivalent to \( (I) \subseteq I \). If \( A \) and \( B \) are non-empty subsets of \( S \), then we have

1. \( A \subseteq (A) \);
2. \( ([A]) = (A) \);
3. If \( A \subseteq B \), then \((A) \subseteq (B)\);
4. \( (A) \Gamma (B) \subseteq (A \Gamma B) \);
5. \( ((A) \Gamma (B)) = (A \Gamma B) \).
Lemma 1. If $I$ and $J$ are $\Gamma$-hyperideals of an ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$, then $I \cap J$ is a $\Gamma$-hyperideal of $S$.

Proof. Let $x \in I$, $y \in J$ and $\gamma \in \Gamma$. Then, $x\gamma y \subseteq I\gamma y \subseteq I\Gamma y \subseteq I$ and $x\gamma y \subseteq I\gamma y \subseteq I\Gamma y \subseteq J$. So, $x\gamma y \subseteq I \cap J$ and hence $\emptyset \neq I \cap J \subseteq S$. We have $(I \cap J)\Gamma S \subseteq I \Gamma S \subseteq I$ and $I \Gamma (I \cap J) \subseteq I \Gamma J \subseteq J$. Similarly, $(I \cap J)\Gamma S \subseteq J$ and $ST(I \cap J) \subseteq I$. So, we have $(I \cap J)\Gamma S \subseteq I \cap J$ and $ST(I \cap J) \subseteq I \cap J$. Now, let $x \in I \cap J$, $y \in S$ and $y \leq x$. Since $I$ and $J$ are $\Gamma$-hyperideals of $S$, we obtain $y \in I$ and $y \in J$. Thus, $y \in I \cap J$. This completes the proof. \qed

Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semihypergroup. A subset $A$ of $S$ is called idempotent if $A = (A\Gamma A)$.

Lemma 2. The $\Gamma$-hyperideals of an ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$ are idempotent if and only if for any $\Gamma$-hyperideals $I, J$ of $S$, we have $I \cap J = (I\Gamma J)$.

Proof. The sufficiency is obvious. For the necessity, let $I, J$ be $\Gamma$-hyperideals of $S$. We have $(I\Gamma J) \subseteq (I\Gamma S) \subseteq (I) = I$ and $(I\Gamma J) \subseteq (I\Gamma J) \subseteq (J) = J$. So, we have $(I\Gamma J) \subseteq I \cap J$. On the other hand, by Lemma 1, $I \cap J$ is a $\Gamma$-hyperideal of $S$. By assumption, we have $I \cap J = ((I \cap J)\Gamma (I \cap J)) \subseteq (I\Gamma J)$. This completes the proof. \qed

Theorem 1. Let $(S, \Gamma, \leq)$ be a commutative ordered $\Gamma$-semihypergroup. If $I$ is a $\Gamma$-hyperideal of $S$ and $A$ is a non-empty subset of $S$, then $(I : A) = \{x \in S \mid x\gamma a \subseteq I \text{ for all } a \in A \text{ and } \gamma \in \Gamma\}$ is a $\Gamma$-hyperideal of $S$.

Proof. Suppose that $x \in (I : A)$, $s \in S$ and $\delta \in \Gamma$. Then, $x\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. We have $(s\delta x)\gamma a = s\delta(x\gamma a) \subseteq S\Gamma I \subseteq I$. So, we have $s\delta x \subseteq (I : A)$. In the similar way, we obtain $x\delta s \subseteq (I : A)$. Now, let $x \in (I : A)$, $y \in S$ and $y \leq x$. Then, $x\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. Also, we have $y\gamma a \leq x\gamma a$ for all $a \in A$ and $\gamma \in \Gamma$, by hypothesis. So, for any $u \in y\gamma a$, $u \leq v$ for some $v \in x\gamma a \subseteq I$. Since $I$ is a $\Gamma$-hyperideal of $S$, it follows that $u \in I$. So, we have $y\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. Thus, we have $y \in (I : A)$. Therefore, $(I : A)$ is a $\Gamma$-hyperideal of $S$. \qed

3 Bi-$\Gamma$-hyperideals

The study of ordered semihyperrings was first undertaken by Davvaz and Omidi [14]. In [35], Omidi, Davvaz and Corsini studied some properties of hyperideals in ordered Krasner hyperrings. The concept of a bi-ideal is a very interesting and important thing in semigroups and ordered semigroups. In 1952, Good and Hughes [18] introduced the notion of bi-ideals in semigroups. Recently, Davvaz et al. [4] introduced the notion of bi-$\Gamma$-hyperideal in $\Gamma$-semihypergroups (cf. [3]). In [36], Pibaljommee and Davvaz studied the properties of bi-hyperideals in ordered semihypergroups. The concept of bi-$\Gamma$-hyperideals of an ordered $\Gamma$-semihypergroup is a generalization of the concept of $\Gamma$-hyperideals (left $\Gamma$-hyperideals, right $\Gamma$-hyperideals) of an ordered $\Gamma$-semihypergroup. First, we define the concept of a bi-$\Gamma$-hyperideal in ordered $\Gamma$-semihypergroups.

Definition 4 ([30]). A sub $\Gamma$-semihypergroup $B$ of an ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$ is called a bi-$\Gamma$-hyperideal of $S$ if the following conditions hold:

1. $B\Gamma S \Gamma B \subseteq B$;
2. When $x \in B$ and $y \in S$ such that $y \leq x$, imply that $y \in B$. 


The concept of bi-$\Gamma$-hyperideals of an ordered $\Gamma$-semihypergroup is a generalization of the concept of $\Gamma$-hyperideals (left $\Gamma$-hyperideals, right $\Gamma$-hyperideals) of an ordered $\Gamma$-semihypergroup. Obviously, every left (right) $\Gamma$-hyperideal of an ordered $\Gamma$-semihypergroup $S$ is a bi-$\Gamma$-hyperideal of $S$, but the following example shows that the converse is not true in general case. Indeed, if $I$ is a left (right) $\Gamma$-hyperideal of $S$, then $I\Gamma \subseteq S\Gamma \subseteq I$. Hence, $I$ is a sub $\Gamma$-semihypergroup of $S$.

**Example 2.** Let $S = \{a, b, c, d, e, f\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows.

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Then $S$ is a $\Gamma$-semihypergroup [41]. We have $(S, \Gamma, \leq)$ is an ordered $\Gamma$-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (a, e), (a, f), (b, b), (c, c), (d, d), (e, e), (f, f)\}.$$

The covering relation and the figure of $S$ are given by:

$$\prec := \{(a, b), (a, c), (a, d), (a, e), (a, f)\}.$$

Here,

1. It is a routine matter to verify that $B_1 = \{a, b, c\}$ is a bi-$\Gamma$-hyperideal of $S$, but it is not a $\Gamma$-hyperideal of $S$.

2. With a small amount of effort one can verify that $B_2 = \{a, b, c, f\}$ is a bi-$\Gamma$-hyperideal of $S$, but it is not a left $\Gamma$-hyperideal of $S$.

**Lemma 3.** The intersection of any family of bi-$\Gamma$-hyperideals of an ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$ is a bi-$\Gamma$-hyperideal of $S$.

**Proof.** Let $\{B_k \mid k \in \Lambda\}$ be a family of bi-$\Gamma$-hyperideals of $S$ and $B = \bigcap_{k \in \Lambda} B_k$. It is easy to check that $B$ is a sub $\Gamma$-semihypergroup of $S$. Now, let $x \in B \Gamma S \Gamma B$. Then, $x \in aas\beta b$ for some $a, b \in B$, $s \in S$ and $a, \beta \in \Gamma$. Since each $B_k$ is a bi-$\Gamma$-hyperideal of $S$, it follows that $aas\beta b \subseteq B_k \Gamma S \Gamma B_k \subseteq B_k$ for all $k \in \Lambda$. Then, $x \in B_k$ for all $k \in \Lambda$. So, we have $x \in \bigcap_{k \in \Lambda} B_k = B$.

Since $x$ was chosen arbitrarily, we have $B \Gamma S \Gamma B \subseteq B$. If $x \in B$ and $y \in S$ such that $y \leq x$, then $x \in B_k$ for all $k \in \Lambda$. Since each $B_k$ is a bi-$\Gamma$-hyperideal of $S$, it follows that $y \in B_k$ for all $k \in \Lambda$. So, we have $y \in \bigcap_{k \in \Lambda} B_k = B$. Hence, $B$ is a bi-$\Gamma$-hyperideal of $S$. $\square$
Lemma 4. Let \((S, \Gamma, \leq)\) be an ordered \(\Gamma\)-semihypergroup. If \(B\) is a bi-\(\Gamma\)-hyperideal of \(S\) and \(C\) is a bi-\(\Gamma\)-hyperideal of \(B\), such that \(C = (C\Gamma)\), then \(C\) is a bi-\(\Gamma\)-hyperideal of \(S\).

**Proof.** By assumption, we have that
\[
\Gamma S \subseteq (\Gamma S \Gamma) \subseteq \Gamma S C \subseteq S C = C,
\]
which shows that \(C\) is a sub \(\Gamma\)-semihypergroup of \(S\). On the other hand, we have \(B\Gamma S \subseteq B\Gamma S C \subseteq C\Gamma S C \subseteq C\). Thus, we have
\[
\Gamma S \Gamma C = (\Gamma S \Gamma \Gamma) = (\Gamma S \Gamma) \subseteq (\Gamma S \Gamma) \subseteq (\Gamma S \Gamma) \subseteq (\Gamma S \Gamma) = C.
\]

Now, let \(c \in C\) and \(x \leq c\), where \(x \in S\). Since \(B\) is a bi-\(\Gamma\)-hyperideal of \(S\) and \(C \subseteq B\), we get \(x \in B\). On the other hand, \(C\) is a bi-\(\Gamma\)-hyperideal of \(B\). It follows that \(x \in C\). This completes the proof. \(\square\)

Let \(A\) be a non-empty subset of an ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\). We denote by \(L_S(A)\) (resp. \(R_S(A)\), \(I_S(A)\)) the left (resp. right, two-sided) \(\Gamma\)-hyperideal of \(S\) generated by \(A\).

Lemma 5. If \(A\) is a non-empty subset of an ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\), then the following hold:

1. \(L_S(A) = (A \cup \Gamma A)\);
2. \(R_S(A) = (A \cup A \Gamma)\);
3. \(I_S(A) = (A \cup \Gamma A \cup A \Gamma \cup \Gamma A \Gamma)\).

**Proof.** Since \(A \subseteq L_S(A)\) and \(\Gamma A \subseteq L_S(A)\), it follows that \((A \cup \Gamma A) \subseteq L_S(A)\). Clearly, \((A \cup \Gamma A) \neq \emptyset\). We have
\[
\Gamma (A \cup \Gamma A) = (\Gamma A \cup \Gamma A) \subseteq (\Gamma A \cup \Gamma A) = (\Gamma A \cup \Gamma A) \subseteq (\Gamma A \cup \Gamma A).
\]

Thus, \((A \cup \Gamma A)\) is a left \(\Gamma\)-hyperideal of \(S\) containing \(A\). This means that \(L_S(A) \subseteq (A \cup \Gamma A)\). This proves that (1) holds. The conditions (2) and (3) are proved similarly. \(\square\)

**Corollary 1.** Let \(a\) be an element of an ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\). Then,

1. \(L_S(a) = (a \cup \Gamma a)\);
2. \(R_S(a) = (a \cup a \Gamma)\);
3. \(I_S(a) = (a \cup \Gamma a \cup a \Gamma \cup \Gamma a \Gamma)\).

Let \(A\) be a non-empty subset of an ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\). We define
\[
\Theta = \{B \mid B \text{ is a bi-}\Gamma\text{-hyperideal of } S \text{ containing } A\}.
\]
Since \(S \in \Theta\), it follows that \(\Theta \neq \emptyset\). We denote by \(B_S(A)\) the bi-\(\Gamma\)-hyperideal of \(S\) generated by \(A\). Clearly, \(A \subseteq B_S(A) = \bigcap_{B \in \Theta} B\). By Lemma 3, \(B_S(A)\) is a bi-\(\Gamma\)-hyperideal of \(S\).
Lemma 6. Let $A$ be a non-empty subset of an ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$. Then,

$$B_S(A) = (A \cup A\Gamma A \cup A\Gamma S \Gamma A).$$

Proof. Set $B = (A \cup A\Gamma A \cup A\Gamma S \Gamma A)$. Clearly, $B \neq \emptyset$. We have

$$B \Gamma B = (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \Gamma (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \subseteq (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \Gamma (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \subseteq (A \Gamma S \Gamma A) \subseteq (A \cup A\Gamma A \cup A\Gamma S \Gamma A).$$

Hence, $B$ is a sub $\Gamma$-semihypergroup of $S$. Now,

$$B \Gamma S \Gamma B = (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \Gamma S (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \subseteq (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \Gamma S (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \subseteq (A \Gamma A \cup A\Gamma S \Gamma A) \subseteq (A \cup A\Gamma A \cup A\Gamma S \Gamma A).$$

Therefore, $B$ is a bi-$\Gamma$-hyperideal of $S$, and hence $B_S(A) \subseteq (A \cup A\Gamma A \cup A\Gamma S \Gamma A)$. Let $C$ be a bi-$\Gamma$-hyperideal of $S$ containing $A$. Then, $A\Gamma A \subseteq C$ and $A\Gamma S \Gamma A \subseteq C\Gamma S \Gamma C \subseteq C$. Thus, we have

$$B = (A \cup A\Gamma A \cup A\Gamma S \Gamma A) \subseteq (C) = C.$$ Hence, $B$ is the smallest bi-$\Gamma$-hyperideal of $S$ containing $A$. Therefore, $B_S(A) = B = (A \cup A\Gamma A \cup A\Gamma S \Gamma A).$ $\square$

Corollary 2. Let $a$ be an element of an ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$. Then,

$$B_S(a) = (a \cup a\Gamma a \cup a\Gamma S \Gamma a).$$

4 MAIN RESULTS

The concepts of regular (resp. intra-regular) ordered $\Gamma$-semihypergroups generalize the corresponding concepts of regular (resp. intra-regular) $\Gamma$-semihypergroups as each regular (resp. intra-regular) $\Gamma$-semihypergroup endowed with the order $\leq := \{(a, b) \mid a = b\}$ is a regular (resp. intra-regular) ordered $\Gamma$-semihypergroup. In this section, we introduce the notion of regular ordered $\Gamma$-semihypergroups and investigate some related results. We characterize regular ordered $\Gamma$-semihypergroups in terms of bi-$\Gamma$-hyperideals, left $\Gamma$-hyperideals and right $\Gamma$-hyperideals of ordered $\Gamma$-semihypergroups. In this paper, some well known results of ordered semihypergroups in case of ordered $\Gamma$-semihypergroups are examined.

Definition 5. An ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$ is called regular if for every $a \in S$ there exist $x \in S, a, b \in \Gamma$ such that $a \leq axb \alpha$. This is equivalent to saying that $a \in (a\Gamma S \Gamma a)$, for every $a \in S$ or $A \subseteq (\Gamma S \Gamma A)$, for every $A \subseteq S$.

Example 3. Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows.

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<tr>
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</table>
Then $S$ is a $\Gamma$-semihypergroup [42]. We have $(S, \Gamma, \leq)$ is an ordered $\Gamma$-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a, a), (a, b), (a, c), (a, e), (b, b), (b, c), (b, e), (c, c), (d, d), (e, e)\}.$$ 

The covering relation and the figure of $S$ are given by:

$$\preceq := \{(a, b), (b, e), (d, c), (e, c)\}.$$ 

We can easily verify that $S$ is a regular ordered $\Gamma$-semihypergroup.

**Lemma 7.** Every $\Gamma$-hyperideal $I$ of a regular ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$ is a regular sub $\Gamma$-semihypergroup of $S$.

**Proof.** Let $a \in I$. Since $S$ is a regular ordered $\Gamma$-semihypergroup, there exist $x \in S$, $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a \leq \alpha a x \beta a \leq \alpha a x \beta a \gamma x \delta a = \alpha (x \beta \gamma x) \delta a$. Since $I$ is a $\Gamma$-hyperideal of $S$, it follows that $x \beta \gamma x \subseteq \Gamma I S \subseteq I$. Thus, $a \leq t$ for some $t \in \alpha a x \beta \gamma x \delta a \subseteq a \Gamma I a$. So, we have $a \in (a \Gamma I a)$. Therefore, $I$ is a regular sub $\Gamma$-semihypergroup of $S$. \hfill $\square$

**Theorem 2.** If $I$ and $J$ are regular $\Gamma$-hyperideals of an ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$, then $I \cap J$ is also a regular $\Gamma$-hyperideal of $S$.

**Proof.** Let $I$ and $J$ are regular $\Gamma$-hyperideals of $S$. By Lemma 1, $I \cap J$ is a $\Gamma$-hyperideal of $S$. By Lemma 7, $I$ and $J$ are regular sub $\Gamma$-semihypergroups of $S$. Now, let $a \in I \cap J$. Then, $a \leq \alpha a x \beta a$ and $a \leq \alpha a x \beta a \gamma \delta a$ for some $x, y \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. So, we have $a \leq \alpha a x \beta a \leq \alpha a x \beta a \gamma \delta a = \alpha (x \beta \mu \alpha x \gamma \delta a)$. Since $I$ and $J$ are $\Gamma$-hyperideals of $S$, we obtain $x \beta \mu \alpha x \gamma \delta a \subseteq I \cap J$. Thus, we have $a \leq t$ for some $t \in \alpha a x \beta \mu \alpha x \gamma \delta a \subseteq a \Gamma (I \cap J) \Gamma a$ which implies that $a \in (a \Gamma (I \cap J) \Gamma a)$. Hence, there exists $z \in I \cap J$ such that $a \leq a x \beta a$. Therefore, $I \cap J$ is a regular sub $\Gamma$-semihypergroup of $S$. \hfill $\square$

We now prove the following theorem which is the crucial theorem in the establishment of our main theorems.

**Theorem 3.** An ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$ is regular if and only if for every right $\Gamma$-hyperideal $R$ and every left $\Gamma$-hyperideal $L$ of $S$, we have $R \cap L = (R \Gamma L)$.

**Proof.** Let $R$ be a right $\Gamma$-hyperideal and $L$ a left $\Gamma$-hyperideal of $S$. As $R \Gamma L \subseteq \Gamma L \subseteq L$ and $R \Gamma L \subseteq \Gamma L \subseteq R$, we have $R \Gamma L \subseteq R \cap L$. So, $(R \Gamma L) \subseteq (R \cap L) \subseteq (R \cap L) \subseteq R \cap L$. Let $S$ be regular, we need to prove that $R \cap L \subseteq (R \Gamma L)$. Since $S$ is regular, we have

$$R \cap L \subseteq ((R \cap L) \Gamma S \Gamma (R \cap L)) \subseteq (R \Gamma S \Gamma (R \cap L)) \subseteq (R \Gamma S \Gamma L) \subseteq (R \Gamma L).$$

Thus, we have $R \cap L = (R \Gamma L)$.\hfill $\square$
Conversely, suppose that \( R \cap L = (RG\Gamma) \) for any right \( \Gamma \)-hyperideal \( R \) and any left \( \Gamma \)-hyperideal \( L \) of \( S \). Let \( a \in S \). Since \( a \in R_S(a) \) and \( a \in L_S(a) \), it follows that \( a \in R_S(a) \cap L_S(a) \). By hypothesis, we have that

\[
a \in (R_S(a) \Gamma L_S(a)) = ((a \cup a \Gamma S) \Gamma (a \cup a \Gamma S) \Gamma) \\
\subseteq (a \Gamma a \cup a \Gamma S a \cup a \Gamma S \Gamma S a) \subseteq (a \Gamma a \cup a \Gamma S \Gamma a).
\]

Hence, \( a \leq t \) for some \( t \in a \Gamma a \cup a \Gamma S a \). If \( u \in a \Gamma S \Gamma a \), then \( a \leq \beta \alpha \beta a \) for some \( x \in S, \beta, \alpha \in \Gamma \). Thus, we have \( a \in (a \Gamma S \Gamma a) \). Therefore, \( S \) is a regular ordered \( \Gamma \)-semihypergroup. If \( u \in a \Gamma a \), then \( a \leq \alpha a \leq a(\alpha \beta a) \). So, we have \( a \in (a \Gamma S \Gamma a) \). Therefore, \( S \) is regular.

Now, we obtain the following corollaries.

**Corollary 3.** If \((S, \Gamma, \leq)\) is a regular ordered \( \Gamma \)-semihypergroup, then \( S = (SG\Gamma) \).

**Corollary 4.** An ordered \( \Gamma \)-semihypergroup \( S \) is called fully \( \Gamma \)-hyperidempotent if every \( \Gamma \)-hyperideal of \( S \) is idempotent. If \( S \) is a regular ordered \( \Gamma \)-semihypergroup, then \( S \) is fully \( \Gamma \)-hyperidempotent.

**Theorem 4.** Let \((S, \Gamma, \leq)\) be a regular ordered \( \Gamma \)-semihypergroup. Then, \( B \) is a bi-\( \Gamma \)-hyperideal of \( S \) if and only if there exists a right \( \Gamma \)-hyperideal \( R \) and a left \( \Gamma \)-hyperideal \( L \) of \( S \) such that \( B = (RG\Gamma L) \).

**Proof.** Let \( S \) be a regular ordered \( \Gamma \)-semihypergroup and \( B \) a bi-\( \Gamma \)-hyperideal of \( S \). First, we show that \((B \Gamma S)\) is a right \( \Gamma \)-hyperideal of \( S \). Let \( y \in S \) and \( x \in (B \Gamma S) \). Then, there exist \( b \in (B \Gamma S) \), \( c \in B \), \( S \in S \) and \( a \in \Gamma \) such that \( x \leq b \leq c \ast \). Since \( S \) is an ordered \( \Gamma \)-semihypergroup, it follows that \( x \beta y \leq b \beta y \leq b \leq (c \ast) \beta y \subseteq B \Gamma S \), where \( \beta \in \Gamma \). Hence, \( x \beta y \subseteq (B \Gamma S) \). If \( y \leq x \), then \( y \leq x \leq b \), and so \( y \in (B \Gamma S) \). Therefore, \((B \Gamma S)\) is a right \( \Gamma \)-hyperideal of \( S \). Similarly, we can prove that \((S \Gamma B)\) is a left \( \Gamma \)-hyperideal of \( S \). Now, we prove that \( B = ((B \Gamma S) \Gamma (S \Gamma B)) \). Since \( S \) is regular, it follows that \( B \subseteq (B \Gamma S \Gamma B) \), for every \( B \subseteq S \). Since \( B \) is a bi-\( \Gamma \)-hyperideal of \( S \), it follows that \( B \Gamma S \Gamma B \subseteq B \). So, we have \((B \Gamma S \Gamma B) \subseteq (B) = B \). Hence, \( B = (B \Gamma S \Gamma B) \). By Corollary 3, we have \( S = (SG\Gamma) \). Hence,

\[
B = (B \Gamma S \Gamma B) = (B \Gamma (SG\Gamma) \Gamma B) = ((B \Gamma (SG\Gamma) \Gamma B) = ((B \Gamma S \Gamma S \Gamma) \Gamma (B)) = ((B \Gamma S \Gamma S \Gamma) \Gamma (B)) = ((B \Gamma S \Gamma S \Gamma) \Gamma (B)) = ((B \Gamma S \Gamma S \Gamma) \Gamma (S \Gamma B)).
\]

Conversely, suppose that \( R \) is a right \( \Gamma \)-hyperideal and \( L \) a left \( \Gamma \)-hyperideal of \( S \) such that \( B = (RG\Gamma L) \). We prove that \((RG\Gamma L)\) is a bi-\( \Gamma \)-hyperideal of \( S \). We have

\[
(RG\Gamma L) \Gamma (RG\Gamma L) \subseteq ((RG\Gamma L) \Gamma) (RG\Gamma L) \subseteq ((RG\Gamma L) \Gamma L) \subseteq (RG\Gamma L) \subseteq (RG\Gamma L) \subseteq (RG\Gamma L) \subseteq (RG\Gamma L).
\]

Then, \((RG\Gamma L)\) is a sub \( \Gamma \)-semihypergroup of \( S \). Also, we have

\[
(RG\Gamma L) \Gamma (RG\Gamma L) = (RG\Gamma L) \Gamma (S) \Gamma (RG\Gamma L) \subseteq ((RG\Gamma L) \Gamma S) \Gamma (RG\Gamma L) \subseteq ((RG\Gamma L) \Gamma S (RG\Gamma L)) \subseteq (RG\Gamma L) \Gamma (RG\Gamma L) \subseteq (RG\Gamma L) \Gamma (RG\Gamma L) \subseteq (RG\Gamma L) \Gamma (RG\Gamma L) \subseteq (RG\Gamma L) \Gamma (RG\Gamma L) \subseteq (RG\Gamma L).
\]

Now, suppose that \( y \in S \) and \( x \in (RG\Gamma L) \) such that \( y \leq x \). Since \( x \in (RG\Gamma L) \), it follows that \( x \leq a \) for some \( a \in RG\Gamma L \). Since \( y \leq x \) and \( x \leq a \), we get \( y \leq a \). So, we have \( y \in (RG\Gamma L) \). Therefore, \((RG\Gamma L)\) is a bi-\( \Gamma \)-hyperideal of \( S \).  

\[\square\]
Theorem 5. An ordered $\Gamma$-semihypergroup $(S, \Gamma, \leq)$ is regular if and only if for every right $\Gamma$-hyperideal $R$, every left $\Gamma$-hyperideal $L$ and every bi-$\Gamma$-hyperideal $B$ of $S$, we have $R \cap B \cap L \subseteq (R \Gamma B \Gamma L)$.

Proof. Let $R$ be right $\Gamma$-hyperideal, $L$ a left $\Gamma$-hyperideal and $B$ a bi-$\Gamma$-hyperideal of $S$. By hypothesis, we have

$$R \cap B \cap L \subseteq ((R \cap B \cap L) \Gamma S (R \cap B \cap L)) \Gamma S ((R \cap B \cap L) \Gamma S (R \cap B \cap L)) \subseteq (R \Gamma S B \Gamma S B \Gamma S L) = ((R \Gamma S) \Gamma (B \Gamma S B) \Gamma (S \Gamma L)) \subseteq (R \Gamma B \Gamma L).$$

Conversely, suppose that $R \cap B \cap L \subseteq (R \Gamma B \Gamma L)$ for every right $\Gamma$-hyperideal $R$, every left $\Gamma$-hyperideal $L$ and every bi-$\Gamma$-hyperideal $B$ of $S$. Since $S$ is a bi-$\Gamma$-hyperideal of $S$, we have $R \cap L = R \cap S \cap L \subseteq (R \Gamma S \Gamma L) \subseteq (R \Gamma L)$. By Theorem 3, $S$ is regular.

Definition 6. Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semihypergroup. An element $a \in S$ is said to be intra-regular if there exist $x, y \in S, a, b, \beta, \gamma \in \Gamma$ such that $a \leq x \alpha y a \gamma$. An ordered $\Gamma$-semihypergroup $S$ is called intra-regular if all elements of $S$ are intra-regular.

Equivalent definitions:

1. $a \in (S \Gamma a \Gamma S)$, for all $a \in S$.
2. $A \subseteq (S \Gamma A \Gamma S)$, for all $A \subseteq S$.

Example 4. Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows.

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<tr>
<th>$\gamma$</th>
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<th>$\beta$</th>
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<td>$a$</td>
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Then $S$ is a $\Gamma$-semihypergroup $[41]$. We have $(S, \Gamma, \leq)$ is an ordered $\Gamma$-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c), (d, d), (e, e)\}.$$

The covering relation and the figure of $S$ are given by:

$$\ll := \{(a, b), (b, c)\}.$$

Then, by routine calculations, $(S, \Gamma, \leq)$ is intra-regular.
Theorem 6. Let \((S, \Gamma, \leq)\) be an ordered \(\Gamma\)-semihypergroup. Then, \(S\) is intra-regular if and only if for every right \(\Gamma\)-hyperideal \(R\) and every left \(\Gamma\)-hyperideal \(L\) of \(S\), we have

\[ R \cap L \subseteq (L \Gamma R). \]

Proof. Let \(R\) be a right \(\Gamma\)-hyperideal and \(L\) a left \(\Gamma\)-hyperideal of \(S\). Let \(S\) be intra-regular; we need to prove that \(R \cap L \subseteq (L \Gamma R)\). Since \(S\) is intra-regular, we have

\[ R \cap L \subseteq (\Sigma(R \cap L) \Gamma (R \cap L) \Gamma S) \subseteq (\Sigma L \Gamma R \Gamma S) \subseteq (L \Gamma R). \]

Conversely, suppose that \(R \cap L \subseteq (L \Gamma R)\) for any right \(\Gamma\)-hyperideal \(R\) and any left \(\Gamma\)-hyperideal \(L\) of \(S\). Let \(a \in S\). Since \(a \in R_S(a)\) and \(a \in L_S(a)\), it follows that \(a \in R_S(a) \cap L_S(a)\). By hypothesis, we have

\[ a \in (L_S(a) \Gamma R_S(a)] = ((a \cup \Sigma a \Gamma a \cup a \Gamma S)] \subseteq (a \Gamma a \cup \Sigma a \Gamma a \Gamma S \cup \Sigma a \Gamma a \Gamma S). \]

Hence, \(a \leq u\) for some \(u \in a \Gamma a \cup \Sigma a \Gamma a \cup a \Gamma S \cup \Sigma a \Gamma a \Gamma S\). If \(u \in \Sigma a \Gamma a \Gamma S\), then \(a \leq x a a x a\gamma y\) for some \(x, y \in S\), \(a, \beta, \gamma \in \Gamma\). Thus, we have \(a \in (\Sigma a \Gamma a \Gamma S)\). Therefore, \(S\) is intra-regular. If \(u \in a \Gamma a\), then \(a \leq a a a \leq a a (a \beta a) \leq a a \beta a a\). So, we have \(a \in (\Sigma a \Gamma a \Gamma S)\). Hence, \(S\) is intra-regular. If \(u \in \Sigma a \Gamma a\), then \(a \leq x a a x a \leq x a (x a a a) a a\) for some \(x \in S\), \(a, \beta, \gamma, \delta \in \Gamma\). So, we have \(a \leq s a \alpha a a\). Hence, \(a \in (\Sigma a \Gamma a \Gamma S)\). If \(u \in a \Gamma a \Gamma S\), in a similar way, we obtain \(a \in (\Sigma a \Gamma a \Gamma S)\). Therefore, \(S\) is intra-regular.

Corollary 5. Let \((S, \Gamma, \leq)\) be an ordered \(\Gamma\)-semihypergroup. Then, the following statements are equivalent:

1. \(S\) is regular and intra-regular.
2. \((R \Gamma L) = R \cap L \subseteq (L \Gamma R)\) for every right \(\Gamma\)-hyperideal \(R\) and every left \(\Gamma\)-hyperideal \(L\) of \(S\).

Proof. It is immediately followed by Theorem 3 and Theorem 6.

Theorem 7. An ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\) is intra-regular if and only if for every right \(\Gamma\)-hyperideal \(R\), every left \(\Gamma\)-hyperideal \(L\) and every bi-\(\Gamma\)-hyperideal \(B\) of \(S\), we have \(R \cap B \cap L \subseteq (L \Gamma B \Gamma R)\).

Proof. The proof is similar to the proof of Theorem 5.

By routine verification we have the following theorem.

Theorem 8. An ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\) is both regular and intra-regular if and only if for every right \(\Gamma\)-hyperideal \(R\), every left \(\Gamma\)-hyperideal \(L\) and every bi-\(\Gamma\)-hyperideal \(B\) of \(S\), we have \(R \cap B \cap L \subseteq (B \Gamma R \Gamma L)\).

Our main aim in the following is to introduce and study the notion of simple ordered \(\Gamma\)-semihypergroups. Also, we characterize this type of ordered \(\Gamma\)-semihypergroups in terms of \(\Gamma\)-hyperideals.

Definition 7. An ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\) is said to be left (resp. right) simple if \(S\) has no proper left (resp. right) \(\Gamma\)-hyperideals. \(S\) is called a simple ordered \(\Gamma\)-semihypergroup if it does not contain proper \(\Gamma\)-hyperideals, i.e., for any \(\Gamma\)-hyperideal \(I \neq \emptyset\) of \(S\), we have \(I = S\).
Lemma 8. Let \( (S, \Gamma, \leq) \) be an ordered \( \Gamma \)-semihypergroup. Then, the following assertions hold:

1. \( S \) is left simple if and only if \( (S\Gamma a) = S \), for all \( a \in S \).
2. \( S \) is right simple if and only if \( (a\Gamma S) = S \), for all \( a \in S \).

Proof. (1): Suppose that \( S \) is a left simple ordered \( \Gamma \)-semihypergroup and \( a \in S \). We have \( S\Gamma(S\Gamma a) = (S\Gamma)(S\Gamma a) \subset (S\Gamma(S\Gamma a)) = ((S\Gamma)(S\Gamma a)) \subset (S\Gamma a) \).

Now, suppose that \( x \in (S\Gamma a) \) and \( y \in S \) such that \( y \leq x \). Since \( x \in (S\Gamma a) \), it follows that \( x \leq u \) for some \( u \in S\Gamma a \). Since \( y \leq x \) and \( x \leq u \), we get \( y \leq u \). So, we have \( y \in (S\Gamma a) \). Hence, \( (S\Gamma a) \) is a left hyperideal of \( S \). Since \( S \) is a left simple ordered \( \Gamma \)-semihypergroup, we have \( (S\Gamma a) = S \).

Conversely, suppose that \( (S\Gamma a) = S \) for all \( a \in S \). Let \( L \) be a left hyperideal of \( S \) and \( x \in L \). By assumption, we have \( (S\Gamma x) = S \). If \( s \in S \), then \( s \in (S\Gamma x) \). So, \( s \leq \delta \) for some \( \delta \in S\Gamma x \subseteq L \). Since \( L \) is a left \( \Gamma \)-hyperideal of \( S \), we have \( s \in L \), and so \( L = S \). Therefore, \( S \) is a left simple ordered \( \Gamma \)-semihypergroup.

(2): The proof is similar to the proof of (1).

Theorem 9. If \( (S, \Gamma, \leq) \) is a left (right) simple ordered \( \Gamma \)-semihypergroup, then \( S \) is a simple ordered \( \Gamma \)-semihypergroup.

Proof. It is straightforward.

Theorem 10. An ordered \( \Gamma \)-semihypergroup \( (S, \Gamma, \leq) \) is left and right simple if and only if for every \( a \in S \), we have \( (S\Gamma a \Gamma S) = S \).

Proof. Let \( S \) be left and right simple and \( a \in S \). By Lemma 8, \( a \in (S\Gamma a) \) and \( a \in (a\Gamma S) \). We have \( a \in (a\Gamma S) \subseteq ((S\Gamma a)\Gamma S) \subseteq (S\Gamma a \Gamma S) \),

and so \( S \subseteq (S\Gamma a \Gamma S) \). Thus, \( (S\Gamma a \Gamma S) = S \).

Conversely, suppose that \( (S\Gamma a \Gamma S) = S \) for every \( a \in S \). Let \( I \) be a \( \Gamma \)-hyperideal of \( S \) such that \( I \nsubseteq S \). Let \( x \in I \). By assumption, we have \( s \leq \mu \gamma \lambda s \) for every \( s \in S \) and \( \mu, \lambda \in \Gamma \). We have \( s \mu \gamma \lambda s \subseteq S \Gamma I S \subseteq (S \Gamma I S) \subseteq (I) = I \).

Then, \( S \nsubseteq I \), a contradiction. Therefore, \( S \) has no proper left and right \( \Gamma \)-hyperideals. This completes the proof.

In what follows, we characterize simple ordered \( \Gamma \)-semihypergroups in terms of bi-\( \Gamma \)-hyperideals.

Theorem 11. An ordered \( \Gamma \)-semihypergroup \( (S, \Gamma, \leq) \) is left and right simple if and only if \( S \) does not contain proper bi-\( \Gamma \)-hyperideals.

Proof. Suppose that \( S \) is a left and right simple ordered \( \Gamma \)-semihypergroup and \( B \) a bi-\( \Gamma \)-hyperideal of \( S \). We claim that \( S \subseteq B \). Consider \( s \in S \) and \( x \in B \). Since \( S \) is left simple, we get \( S = (x \cup S\Gamma x) \). We can consider the following two cases:

Case 1. If \( s \leq x \), then we have \( s \in B \).

Case 2. Let \( s \in (u\gamma x) \) for some \( u \in S \) and \( \gamma \in \Gamma \). By hypothesis, \( S \) is a right simple ordered \( \Gamma \)-semihypergroup. Then, we have \( S = (x \cup S\Gamma x) \). Since \( u \in S \), we have \( u \leq x \) or \( u \in (x \delta w) \) for some \( w \in S \) and \( \delta \in \Gamma \). By Lemma 8, we have \( S = (x\Gamma S) = (S\Gamma x) \), and so \( x \in (x\Gamma S) = (x\Gamma(S\Gamma x)) \subseteq (x\Gamma S\Gamma x) \). Then, \( S \) is a regular ordered \( \Gamma \)-semihypergroup. Thus, there exists \( a \in S \) and \( \alpha, \beta \in \Gamma \) such that \( x \in (x\alpha a\beta x) \). If \( u \leq x \), then we have
\((u\gamma x) \subseteq (x\gamma x) \subseteq (x\gamma x\alpha x) \subseteq (B\Gamma S\Gamma B) \subseteq B,\)

and so \(s \in B.\) If \(u \in (x\delta w),\) then we have
\[(u\gamma x) \subseteq (x\delta w\gamma x) \subseteq (B\Gamma S\Gamma B) \subseteq B,\]
and so \(s \in B.\) Therefore, \(S \subseteq B.\)

Conversely, suppose that \(S\) does not contain proper bi-\(\Gamma\)-hyperideals. Let \(L\) be a left \(\Gamma\)-hyperideal of \(S\). Then, \(L\) is a bi-\(\Gamma\)-hyperideal of \(S\). By assumption, we have \(S = L.\) Therefore, \(S\) is a left simple ordered \(\Gamma\)-semihypergroup. Similarly, we can show that \(S\) is a right simple ordered \(\Gamma\)-semihypergroup. \(\Box\)

In the following, we study some properties of bi-\(\Gamma\)-hyperideals and minimal bi-\(\Gamma\)-hyperideals in ordered \(\Gamma\)-semihypergroups.

**Definition 8.** An ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\) is said to be \(B\)-simple if \(S\) does not contain any proper bi-\(\Gamma\)-hyperideals. A bi-\(\Gamma\)-hyperideal \(C\) of \(S\) is called a minimal bi-\(\Gamma\)-hyperideal of \(S\) if \(C\) does not properly contain any bi-\(\Gamma\)-hyperideal of \(S\).

**Theorem 12.** Let \(B\) be a bi-\(\Gamma\)-hyperideal of an ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\). Then, \((u\Gamma B\Gamma v)\) is a bi-\(\Gamma\)-hyperideal of \(S\) for every \(u, v \in S\). In particular, \((u\Gamma S\Gamma v)\) is a bi-\(\Gamma\)-hyperideal of \(S\) for every \(u, v \in S.\)

*Proof.* The proof is similar to the proof of Theorem 2.2 in [8]. \(\Box\)

**Corollary 6.** Let \((S, \Gamma, \leq)\) be an ordered \(\Gamma\)-semihypergroup. Then, \(S\) is \(B\)-simple if and only if \((u\Gamma S\Gamma u) = S\) for all \(u \in S.\)

*Proof.* The necessity is obvious. For the sufficiency, let \((u\Gamma S\Gamma u) = S\) for all \(u \in S.\) We have
\[
(u\Gamma S\Gamma u) \subseteq (S\Gamma u) \subseteq S \quad \text{and} \quad (u\Gamma S\Gamma u) \subseteq (u\Gamma S) \subseteq S.
\]
By assumption, we have \((S\Gamma u) = S\) and \((u\Gamma S) = S\) for all \(u \in S.\) Now, let \(B\) be a bi-\(\Gamma\)-hyperideal of \(S\) and \(b \in B.\) Then, \((S\Gamma b) = S = (b\Gamma S).\) So, we have
\[
S = (b\Gamma S) = (b\Gamma (b\Gamma S)) \subseteq (b\Gamma S\Gamma b) \subseteq (B\Gamma S\Gamma B) \subseteq (B) \subseteq B.
\]
This completes the proof. \(\Box\)

**Corollary 7.** Let \((S, \Gamma, \leq)\) be an ordered \(\Gamma\)-semihypergroup. If \(C\) is a minimal bi-\(\Gamma\)-hyperideal of \(S\) and \(B\) a bi-\(\Gamma\)-hyperideal of \(S\), then \(C = (c\Gamma B\Gamma d)\) for every \(c, d \in C.\)

*Proof.* By Theorem 12, \((c\Gamma B\Gamma d)\) is a bi-\(\Gamma\)-hyperideal of \(S.\) Since \(C\) is a minimal bi-\(\Gamma\)-hyperideal of \(S\) and \((c\Gamma B\Gamma d) \subseteq (C\Gamma B\Gamma C) \subseteq (C\Gamma S\Gamma C) \subseteq (C) \subseteq C,\) we obtain \(C = (c\Gamma B\Gamma d).\) \(\Box\)

At the end of the paper, we prove the following theorem.

**Theorem 13.** Let \(B\) be a bi-\(\Gamma\)-hyperideal of an ordered \(\Gamma\)-semihypergroup \((S, \Gamma, \leq)\). Then, \(B\) is a minimal bi-\(\Gamma\)-hyperideal of \(S\) if and only if \(B\) is \(B\)-simple.
Hence, $C$ is a bi-$\Gamma$-hyperideal of $S$. Then, $B$ is a sub $\Gamma$-semihypergroup of $S$. Now, let $C$ be a bi-$\Gamma$-hyperideal of $B$. Thus, $\text{CTBG}_{C} \subseteq C$. Put $K = (\text{CTBG}_{C})_{C}$. Then, $\emptyset \neq K \subseteq C \subseteq B$. Now, we prove that $K$ is a bi-$\Gamma$-hyperideal of $S$. Let $k_1, k_2 \in K$, $x \in S$ and $\gamma, \delta \in \Gamma$. Then, $k_1 \leq c_1 a_1 b_1 \beta_1 c_1'$ and $k_2 \leq c_2 a_2 b_2 \beta_2 c_2'$ for some $c_1, c_1', c_2, c_2' \in C$, $b_1, b_2 \in B$ and $a_1, \beta_1, a_2, \beta_2 \in \Gamma$. So, we have

$$k_1 \gamma k_2 \leq c_1 a_1 (b_1 \beta_1 c_1' \gamma c_2 a_2 b_2) \beta_2 c_2'.$$

and

$$k_1 \gamma \delta k_2 \leq c_1 a_1 (b_1 \beta_1 c_1' \gamma \delta c_2 a_2 b_2) \beta_2 c_2'.
$$

Since $b_1 \beta_1 c_1' \gamma c_2 a_2 b_2 \subseteq \text{BGST}_B \subseteq B$, it follows that $k_1 \gamma k_2 \subseteq \text{KTK} \subseteq \text{CTC} \subseteq C$. So, $k_1 \gamma k_2 \subseteq (\text{CTBG}_{C})_{C} = K$. Hence, $K$ is a sub $\Gamma$-semihypergroup of $S$. Since $b_1 \beta_1 c_1' \gamma \delta c_2 a_2 b_2 \subseteq \text{BGST}_B \subseteq B$, we get

$$c_1 a_1 (b_1 \beta_1 c_1' \gamma \delta c_2 a_2 b_2) \beta_2 c_2' \subseteq \text{CTBG}_{C} \subseteq C.$$

Since $C$ is a bi-$\Gamma$-hyperideal of $B$ and $k_1 \gamma \delta k_2 \subseteq \text{KTK} \subseteq \text{BGST}_B \subseteq B$, we obtain $k_1 \gamma \delta k_2 \subseteq C$. So, we have $k_1 \gamma \delta k_2 \subseteq (\text{CTBG}_{C})_{C} = K$. Therefore, $\text{KTK} \subseteq K$. Now, let $y \in (K)$. Then, $y \leq k$ for some $k \in K$. Since $k \in K$, there exist $c, c' \in C$, $b \in B$ and $\mu, \lambda \in \Gamma$ such that $k \leq c \mu b \lambda c'$. Since $c \mu b \lambda c' \subseteq \text{CTBG}_{C} \subseteq C \subseteq B$ and $B$ is a bi-$\Gamma$-hyperideal of $S$, we get $k \in B$. Since $B$ is a bi-$\Gamma$-hyperideal of $S$, we have $y \in B$. So, $y \leq z$ for some $z \in c \mu b \lambda c' \subseteq \text{CTBG}_{C} \subseteq C$. Since $C$ is a bi-$\Gamma$-hyperideal of $B$, we have $y \in C$. So, we have $y \in (\text{CTBG}_{C})_{C} = K$. Therefore, $K$ is a bi-$\Gamma$-hyperideal of $S$. Since $B$ is a minimal bi-$\Gamma$-hyperideal of $S$, it follows that $K = B$. So, we have $C = B$. Therefore, $B$ is $B$-simple.

Conversely, assume that $B$ is $B$-simple. Let $C$ be a bi-$\Gamma$-hyperideal of $S$ such that $C \subseteq B$. Then, $B \cap C \neq \emptyset$. Let $c \in B \cap C$. By Theorem 12, $(c \text{BG}_{C})$ is a bi-$\Gamma$-hyperideal of $B$. Since $B$ is $B$-simple, we obtain $(c \text{BG}_{C}) = B$. Now, we have

$$B = (c \text{BG}_{C}) \subseteq (\text{CTBG}_{C}) \subseteq (\text{CTSC}) \subseteq (C) = C.$$

Hence, $C = B$. Therefore, $B$ is a minimal bi-$\Gamma$-hyperideal of $S$. 

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Поняття $\Gamma$-напівгіпергруп є узагальненням напівгруп, узагальненням напівгіпергруп і узагальненням $\Gamma$-напівгруп. У даній роботі досліджується поняття bi-$\Gamma$-гіперідеалів у впорядкованих $\Gamma$-напівгіпергрупах і досліджуються деякі властивості цих bi-$\Gamma$-гіперідеалів. Також ми визначаємо і використовуємо поняття регулярно впорядкованих $\Gamma$-напівгіпергруп для вивчення деяких класичних результатів і властивостей у впорядкованих $\Gamma$-напівгіпергрупах.

Ключові слова і фрази: упорядковані $\Gamma$-напівгіпергрупи, Г-гіперідеали, bi-Г-гіперідеали.