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ON $\varphi$-SYMMETRIC $\tau$-CURVATURE TENSOR IN $N(k)$-CONTACT METRIC MANIFOLD

In this paper we study $\tau$-curvature tensor in $N(k)$-contact metric manifold. We study $\tau$-$\varphi$-recurrent, $\tau$-$\varphi$-symmetric and globally $\tau$-$\varphi$-symmetric $N(k)$-contact metric manifold.

Key words and phrases: contact metric manifold, symmetry.

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INTRODUCTION

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. In the context of contact geometry the notion of $\varphi$-symmetry is introduced and studied by Boeckx E., Buecken P. and Vanhecke L. [3] with several examples. As a weaker version of local symmetry, Takahashi T. [13] introduced the notion of locally $\varphi$-symmetry on a Sasakian manifold. Generalizing the notion of $\varphi$-symmetry, the authors De U.C., Shaikh A.A. and Sudipta Biswas [4] introduced the notion of $\varphi$-recurrent Sasakian manifolds. This notion has been studied by many authors for different types of contact manifolds like Venkatesha and Bagewadi C.S. [14, 15], De U.C. and Abdul Kalam Gazi [5], Nagaraja H.G. [9] etc.

In [12] Tanno S. introduced the notion of $k$-nullity distribution of a contact metric manifold as a distribution such that the characteristic vector field $\xi$ of the contact metric manifold belongs to the distribution. The contact metric manifold with $\xi$ belonging to the $k$-nullity distribution is called $N(k)$-contact metric manifold such a manifold is also studied by various authors. Generalizing this notion in 1995, Blair D.E., Koufogiorgos T. and Papantoniou B.J. [2] introduced the notion of a contact metric manifold with $\xi$ belonging to the $(k, \mu)$-nullity distribution, where $k$ and $\mu$ are real constants. In particular, if $\mu = 0$ then the notion of $(k, \mu)$-nullity distribution reduces to the notion of $k$-nullity distribution.

In [6] Mukut Mani Tripathi and et.al. introduced the $\tau$-curvature tensor which is a particular cases of known curvatures like conformal, concircular, projective, $M$-projective, $W_i$-curvature tensor ($i = 0, \ldots, 9$) and $W_j^*$-curvature tensor ($j = 0, 1$). Further, in [7, 8] Mukut Mani Tripathi and et.al. studied $\tau$-curvature tensor in K-contact, Sasakian and Semi-Riemannian manifolds. Later in [10] the authors studied some properties of $\tau$-curvature tensor and they obtained some interesting results.

Motivated by all these work in this paper we studied the $\varphi$-symmetric $\tau$-curvature tensor in $N(k)$-contact metric manifold.

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1 Preliminaries

A $(2n + 1)$-dimensional differential manifold $M$ is said to have an almost contact structure $(\varphi, \xi, \eta)$ if it carries a tensor field $\varphi$ of type $(1,1)$, a vector field $\xi$ and 1-form $\eta$ satisfying
\[ \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0. \tag{1} \]

Let $g$ be a compatible Riemannian metric with almost contact structure $(\varphi, \xi, \eta)$ such that,
\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \]
\[ g(\varphi X, Y) = -g(X, \varphi Y), \quad g(X, \xi) = \eta(X). \tag{2} \]

where $X, Y$ are vector fields defined on $M$. Then the structure $(\varphi, \xi, \eta, g)$ on $M$ is said to have an almost contact metric structure and the manifold $M$ equipped with this structure is called an almost contact metric manifold [1]. An almost contact metric structure $(\varphi, \xi, \eta, g)$ becomes a contact metric structure if for all vector fields $X, Y$ on $M$ we have $d\eta(X, Y) = g(X, \varphi Y)$.

Given a contact metric manifold $(M, \varphi, \xi, \eta, g)$, we define a $(1, 1)$ tensor field $h$ by $h = \frac{1}{2}L\varphi$, where $L$ denotes the Lie differentiation. Then $h$ is symmetric and satisfies $h\varphi = -\varphi h$. Also we have $Tr.h = Tr.Q\varphi = 0$ and $h\xi = 0$. Moreover, if $\nabla$ denotes the Riemannian connection on $M$, then the following relation holds
\[ \nabla X\xi = -\varphi X - \varphi hX. \tag{3} \]

For a contact metric manifold $M(\varphi, \xi, \eta, g)$ the $(k, \mu)$-nullity distribution is
\[ p \rightarrow N_p(k, \mu) = \{ Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}, \]
for all vector fields $X, Y \in T_pM$, where $k, \mu$ are real numbers and $R$ is the curvature tensor. Hence if the characteristic vector field $\xi$ belongs to the $(k, \mu)$-nullity distribution, then we have
\[ R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{4} \]

Thus a contact metric manifold satisfying (4) is called a $(k, \mu)$-contact metric manifold. In particular, if $\mu = 0$, then the notion of $(k, \mu)$-nullity distribution reduces to the notion of $k$-nullity distribution introduced by Tanno S. [12]. In a $(k, \mu)$-contact metric manifold [11] the following relations hold:
\[ h^2 = (k - 1)\varphi^2, \quad k \leq 1, \]
\[ (\nabla X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)[X + hX], \]
\[ R(\xi, X)Y = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)], \]
\[ \eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + \mu[\eta(Y)hX - \eta(X)hY], \]
\[ S(X, \xi) = 2nk\eta(X), \quad S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) \]
\[ + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1, \]
\[ r = 2n[2n - 2 + k - n\mu], \]
\[ S(\varphi X, \varphi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \]

where $S$ is the Ricci tensor of type $(0, 2)$, $Q$ is the Ricci operator, that is, $S(X, Y) = g(QX, Y)$ and $r$ is the scalar curvature of the manifold. From (2), it follows that
\[ (\nabla X\eta)Y = g(X + hX, \varphi Y). \]
The $k$-nullity distribution $N(k)$ of a Riemannian manifold $M$ is defined by
\[ p \to N_p(k) = \{ Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \}, \]
k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold. In $N(k)$-contact metric manifold the following relations hold [5]:

\[ h^2 = (k - 1)\varphi^2, \quad k \leq 1, \]

\[ (\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)[X + hX], \]

\[ R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X], \]

\[ S(X, \xi) = 2nk\eta(X), \]

\[ S(\xi, X)Y = 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y) + [2(1 - n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1, \]

\[ QX = 2(n - 1)X + 2(n - 1)hX + [2(1 - n) + 2nk]\eta(X)\xi, \quad n \geq 1, \]

\[ r = 2n[2n - 2 + k], \]

\[ S(\varphi X, \varphi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n - 1)g(hX, Y), \]

\[ (\nabla_X \eta)Y = g(X + hX, \varphi Y). \]

**Definition 1.** An $N(k)$-contact metric manifold $M$ is said to be locally $\varphi$-symmetric if

\[ \varphi^2((\nabla_W R)(X, Y)Z) = 0 \]

for all vector fields $X, Y, Z, W$, which are orthogonal to $\xi$.

This notion was introduced by Takahashi T. [13] for Sasakian manifolds.

**Definition 2.** An $N(k)$-contact metric manifold $M$ is said to be $\varphi$-symmetric if

\[ \varphi^2((\nabla_W R)(X, Y)Z) = 0 \]

for all arbitrary vector fields $X, Y, Z, W$.

**Definition 3.** An $N(k)$-contact metric manifold $M$ is said to be locally $\tau$-$\varphi$-symmetric if

\[ \varphi^2((\nabla_W \tau)(X, Y)Z) = 0 \]

for all vector fields $X, Y, Z, W$, which are orthogonal to $\xi$.

**Definition 4.** An $N(k)$-contact metric manifold $M$ is said to be $\varphi$-recurrent if and only if there exists a non zero 1-form $A$ such that

\[ \varphi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z \]

for all vector fields $X, Y, Z, W$. Here $X, Y, Z, W$ are arbitrary vector fields which are not necessarily orthogonal to $\xi$.

If the 1-form $A$ vanishes identically, then the manifold is said to be a locally $\varphi$-symmetric manifold.

**Definition 5.** An $N(k)$-contact metric manifold $M$ is said to be $\tau$-$\varphi$-recurrent if and only if there exists a non zero 1-form $A$ such that

\[ \varphi^2((\nabla_W \tau)(X, Y)Z) = A(W)\tau(X, Y)Z, \]

for all vector fields $X, Y, Z, W$. Here $X, Y, Z, W$ are arbitrary vector fields which are not necessarily orthogonal to $\xi$. 

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where \( a_0, \ldots, a_7 \) are some smooth functions on \( M \). For different values of \( a_0, \ldots, a_7 \) the \( \tau \)-curvature tensor reduces to the curvature tensor \( R \), quasi-conformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor, projective curvature tensor, \( M \)-projective curvature tensor, \( W_i \)-curvature tensors \((i = 0, \ldots, 9)\), \( W_i^* \)-curvature tensors \((j = 0, 1)\).

### 2 \( \tau \)-\( \varphi \)-RECURRENT \( N(k) \)-CONTACT METRIC MANIFOLD

In this section, we define \( \tau \)-\( \varphi \)-recurrent \( N(k) \)-contact metric manifold by

\[
\varphi^2((\nabla_W \tau)(X, Y)Z) = A(W)\tau(X, Y)Z
\]

for all vector fields \( X, Y, Z, W \). By virtue of (1), we have

\[
-(\nabla_W \tau)(X, Y)Z + \eta((\nabla_W \tau)(X, Y)Z)\xi = A(W)\tau(X, Y)Z.
\]

By taking an inner product with \( U \), then we get

\[
-g((\nabla_W \tau)(X, Y)Z, U) + \eta((\nabla_W \tau)(X, Y)Z)g(\xi, U) = A(W)g(\tau(X, Y)Z, U).
\]

Let \( \{e_i : i = 1, 2, \ldots, 2n + 1\} \) be an orthonormal basis of the tangent space at any point of the manifold. Putting \( X = U = e_i \) in (11) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), by virtue of (11), we obtain

\[
-a_0 + (2n + 1)a_1 + a_2 + a_3(\nabla_W S)(Y, Z) - [a_4 + 2na_7](\nabla_W r)g(Y, Z)
\]

\[
-a_5g((\nabla_W Q)Y, Z) - a_6g((\nabla_W Q)Z, Y) + a_0\eta((\nabla_W R)(\xi, Y)Z) + a_1(\nabla_W S)(Y, Z)
\]

\[
+ a_2(\nabla_W S)(\xi, Z)\eta(Y) + a_3(\nabla_W S)(Y, \xi)\eta(Z) + a_4g(Y, Z)\eta((\nabla_W Q)\xi)
\]

\[
+ a_5\eta(Z)\eta((\nabla_W Q)Y) + a_6\eta(Y)\eta((\nabla_W Q)Z) + a_7(\nabla_W r)[g(Y, Z) - \eta(Y)\eta(Z)]
\]

\[
= A(W)[[a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6]S(Y, Z) + [a_4 + 2na_7]rg(Y, Z)].
\]

Putting \( Z = \xi \) in (12) and simplifying, we get

\[
-a_0 + 2na_1 + a_2 + a_3(\nabla_W S)(Y, \xi) - [a_4 + 2na_7](\nabla_W r)\eta(Y)
\]

\[
-a_6g((\nabla_W Q)\xi, Y) + a_3(\nabla_W S)(Y, \xi)
\]

\[
= A(W)\eta(Y)[a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6]2nk + [a_4 + 2na_7]r.
\]

We know that

\[
(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).
\]

By using (3), (5) in (14), we have

\[
(\nabla_W S)(Y, \xi) = S(Y, \varphi W) + S(Y, \varphi h W) - 2nkg(Y, \varphi W) - 2nkg(Y, \varphi h W).
\]

Substituting (15) in (14), we obtain

\[
-a_0 + 2na_1 + a_2 + a_6[S(Y, \varphi W) + S(Y, \varphi h W) - 2nkg(Y, \varphi W) - 2nkg(Y, \varphi h W)]
\]

\[
-[a_4 + 2na_7](\nabla_W r)\eta(Y)
\]

\[
= [a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6]2nkA(W)\eta(Y) + [a_4 + 2na_7]rA(W)\eta(Y).
\]
Replacing $Y$ by $\phi Y$ in (16), we have
\[ -[a_0 + 2na_1 + a_2 + a_6][S(\phi Y, \phi W) + S(\phi Y, \phi hW)] \\
- 2nk[g(\phi Y, \phi W) + g(\phi Y, \phi hW)] = 0. \] (17)

If $[a_0 + 2na_1 + a_2 + a_6] \neq 0$, then by virtue of (1) and (8) in (17), we obtain
\[ S(Y, W) = [2nk - 2(n - 1)(k - 1)]g(Y, W) + 2(n - 1)(k - 1)\eta(Y)\eta(W) + [2nk + 2(n - 1)(k - 1)]g(Y, hW) - 2nk + 2(n - 1)(k - 1)\eta(Y)\eta(hW). \] (18)

Replacing in place $W$ as $hW$ in (18), we get
\[ g(Y, hW) = n(k - 1)g(Y, W) - n(k - 1)\eta(Y)\eta(W). \] (19)

By substituting (19) in (18), we get
\[ S(Y, W) = [2nk + 2(n - 1)^2(k - 1) + 2n^2k(k - 1)]g(Y, W) + [-2(n - 1)^2(k - 1) - 2n^2k(k - 1)]\eta(Y)\eta(W). \]

Hence we state the following

**Theorem 1.** A $\tau$-$\phi$-recurrent $N(k)$-contact metric manifold is an $\eta$-Einstein manifold with $-\eta = 0$.

Now from (10), we have
\[ (\nabla_W \tau)(X, Y)Z = \eta(\nabla_W \tau)(X, Y)Z - A(W)\tau(X, Y)Z, \] (20)
from (20) and the second Bianchi identity, we get
\[ (\nabla_W \tau)(X, Y)Z + (\nabla_X \tau)(Y, W)Z + (\nabla_Y \tau)(W, X)Z \\
= \eta(\nabla_W \tau)(X, Y)Z \xi + \eta(\nabla_X \tau)(Y, W)Z \xi + \eta(\nabla_Y \tau)(W, X)Z \xi \\
- \{A(W)\tau(X, Y)Z + A(X)\tau(Y, W)Z + A(Y)\tau(W, X)Z\}. \] (21)

From (21), we get
\[ A(W)\eta(\tau(X, Y)Z) + A(X)\eta(\tau(Y, W)Z) + A(Y)\eta(\tau(W, X)Z) = 0. \] (22)

By using (9) in (22), we obtain
\[ A(W)\{a_0k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + a_1\eta(X)[2(n - 1)g(Y, Z) + g(hY, Z)] \\
+ [2(1 - n) + 2nk]\eta(Y)\eta(Z)] + a_2\eta(Y)[2(n - 1)g(X, Z) + g(hX, Z) + [2(1 - n) + 2nk]\eta(X)\eta(Z)] + a_3\eta(Z)[2(n - 1)g(Y, X) + 2n\eta(Y)] \\
+ [2(n - 1)^2 + 2nk]\eta(X)\eta(Y)\eta(Z) + 2nka_4g(Y, Z)\eta(X) + 2nka_5g(X, Z)\eta(Y) \\
+ 2nka_6g(Y, X)\eta(Y) + a_7[g(Y, Z)\eta(Y) + g(X, Z)\eta(Y)]\} + A(X)\{a_0k[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \\
+ a_1\eta(Y)[2(n - 1)g(W, Z) + g(hW, Z) + [2(1 - n) + 2nk]\eta(W)\eta(Z)] + a_2\eta(W)[2(n - 1)g(Y, Z) + g(hY, Z) + [2(1 - n) + 2nk]\eta(Y)\eta(Z)] + a_3\eta(Z)[2(n - 1)g(Y, W) + 2n\eta(Y)] \\
+ [2(n - 1)^2 + 2nk]\eta(Y)\eta(Z) + 2nka_4g(W, Z)\eta(Y) + 2nka_5g(Y, Z)\eta(W) \\
+ 2nka_6g(Y, W)\eta(Z) + a_7[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)]\} + A(Y)\{a_0k[g(X, Z)\eta(W) - g(W, Z)\eta(Y)] \\
+ a_1\eta(Y)[2(n - 1)g(X, Z) + 2n\eta(Y)] + [2(1 - n) + 2nk]\eta(X)\eta(Z)] + a_2\eta(X)[2(n - 1)g(W, Z) + g(hW, Z) + [2(1 - n) + 2nk]\eta(X)\eta(Z)] + a_3\eta(Z)[2(n - 1)g(W, X) + 2n\eta(X)] \\
+ [2(n - 1)^2 + 2nk]\eta(X)\eta(Y) + 2nka_4g(X, Z)\eta(W) + 2nka_5g(W, Z)\eta(X) \\
+ 2nka_6g(W, X)\eta(Y) + a_7[g(X, Z)\eta(W) - g(W, Z)\eta(X)]\} = 0. \] (23)
Putting $Y = Z = e_i$ in (23), we get
\[
A(W)\eta(X)[(2n - 1)(a_0 k + ra_7) + 2na_1[2(n - 1) + k] + [2nk + 2(n - 1)]a_2
+ 2nk[a_3 + (2n + 1)a_4 + 2a_5 + a_6]) + A(X)\eta(W)[-(2n - 1)(a_0 k + ra_7)
+ a_1[2(n - 1) + k] + 2na_2[k + 2(n - 1)] + 2nk[a_3 + 2a_4 + (2n + 1)a_5 + a_6] +
[a_1 + a_2 + a_3][2(1 - n) + 2nk]A(\xi)\eta(X)\eta(W) + 2(n - 1)a_1 A(hX)\eta(W)
+ 2(n - 1)a_2 A(hW)\eta(X) + [2(n - 1)a_3 + 2nka_6]A(\xi)g(W, X)
+ 2(n - 1)a_3 A(\xi)g(hW, X) = 0.
\]

Putting $X = \xi$ in (24) and simplifying, we obtain
\[
A(W)[(2n - 1)(a_0 k + ra_7) + 2na_1[2(n - 1) + k] + a_2[2nk + 2(n - 1)]
+ 2nk[a_3 + (2n + 1)a_4 + 2a_5 + a_6]) + A(\xi)\eta(W)[-(2n - 1)(a_0 k + ra_7)
+ a_2[2(n - 1)(2n - 1) + 4nk] + 4nk[a_1 + a_3 + a_4 + a_6] + 2nk(2n + 1)a_5]
+ 2(n - 1)a_2 A(hW) = 0.
\]

Replacing $W$ by $hW$ in (25), we obtain
\[
A(hW) = \frac{2(n - 1)a_2(k - 1)}{L}[A(W) - A(\xi)\eta(W)],
\]
where
\[
L = (2n - 1)(a_0 k + ra_7) + 2na_1[2(n - 1) + k] + a_2[2nk + 2(n - 1)]
+ 2nk[a_3 + (2n + 1)a_4 + 2a_5 + a_6].
\]

Substituting (26) in (25), we get
\[
A(W) = \frac{-[ML - E]}{L^2 + E}A(\xi)\eta(W),
\]
where
\[
M = -(2n - 1)(a_0 k + ra_7) + a_2[2(n - 1)(2n - 1) + 4nk]
+ 2nk[a_1 + a_3 + 2a_4 + (2n + 1)a_5 + 2a_6],
E = 4(n - 1)^2a_2^2(k - 1).
\]

Here $A(\xi) = g(\xi, \rho)$, $\rho$ being the vector field associated to the 1-form $A$, that is, $g(X, \rho) = A(X)$. Hence we state the following

**Theorem 2.** In a $\tau$-$\varphi$-recurrent $N(k)$-contact metric manifold the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1-form $A$ are codirectional and the 1-form $A$ is given in (27).

### 3 $\tau$-$\varphi$-Symmetric $N(k)$-Contact Metric Manifold

In this section we define $\tau$-$\varphi$-symmetric $N(k)$-contact metric manifold by
\[
\varphi^2((\nabla_W \tau)(X, Y)Z) = 0
\]
for all vector fields $X, Y, Z, W$, which are orthogonal to $\xi$. By using (6), (7) in (9), we get
\[ \tau(X, Y)Z = a_0 R(X, Y)Z + a_1 [2(n - 1)g(Y, Z)X + [2(1 - n) + 2nk] \eta(Y)\eta(Z)X \\
+ 2(n - 1)g(hY, Z)X] + a_2 [2(n - 1)g(X, Z)Y + 2(n - 1)g(hX, Z)Y \\
+ [2(1 - n) + 2nk] \eta(X)\eta(Z)Y] + a_3 [2(n - 1)g(X, Y)Z + 2(n - 1)g(hX, Y)Z \\
+ [2(1 - n) + 2nk] \eta(X)\eta(Y)Z] + a_4 g(Y, Z)[2(1 - n)X + 2(n - 1)hX \\
+ [2(1 - n) + 2nk] \eta(Y)\xi] + a_5 g(X, Z)[2(n - 1)Y + 2(n - 1)hY \\
+ [2(1 - n) + 2nk] \eta(Y)\xi] + a_6 g(X, Y)[2(n - 1)Z + 2(n - 1)hZ \\
+ [2(1 - n) + 2nk] \eta(Y)\xi] + a_7 r[g(Y, Z)X - g(X, Z)Y]. \]

Differentiating (28) with respect to \( W \), we obtain

\[
(\nabla_W \tau)(X, Y)Z = a_0 (\nabla_W R)(X, Y)Z + a_1 [2(n - 1) + 2nk] \{g(Y, \nabla_W \xi)\eta(Z)X \\
+ g(Z, \nabla_W \xi)\eta(Y)X\} + 2(n - 1)g((\nabla_W h)Y, Z)X + a_2 [2(n - 1)g((\nabla_W h)X, Z)Y \\
+ [2(1 - n) + 2nk] \{g(X, \nabla_W \xi)\eta(Z)Y + g(Z, \nabla_W \xi)\eta(X)Y\}] \\
+ a_3 [2(n - 1)g((\nabla_W h)X, Y) + [2(1 - n) + 2nk] \{g(X, \nabla_W \xi)\eta(Y) + g(Y, \nabla_W \xi)\eta(X)\}]Z \\
+ a_4 g(Y, Z)[2(n - 1)(\nabla_W h)X + [2(1 - n) + 2nk] \{g(X, \nabla_W \xi)\eta(Y) + g(Y, \nabla_W \xi)\eta(X)\}] \\
+ a_5 g(X, Z)[2(n - 1)(\nabla_W h)Y + [2(1 - n) + 2nk] \{g(Y, \nabla_W \xi)\eta(X) + g(X, \nabla_W \xi)\eta(Y)\}] \\
+ a_6 g(X, Y)[2(n - 1)(\nabla_W h)Z + [2(1 - n) + 2nk] \{g(Z, \nabla_W \xi)\eta(Y) + g(Y, \nabla_W \xi)\eta(X)\}] \\
+ a_7 (\nabla_W r)[g(Y, Z)X - g(X, Z)Y].
\]

We assume that all vector fields \( X, Y, Z, W \) are orthogonal to \( \xi \), then we have

\[
(\nabla_W \tau)(X, Y)Z = a_0 (\nabla_W R)(X, Y)Z + a_4 g(Y, Z)[2(n - 1) \{(1 - k)g(W, \varphi X) \\
+ g(W, \varphi h X)\} \xi + [2(1 - n) + 2nk] \{-g(X, \varphi h X) - g(X, \varphi h W)\} \xi] \\
+ a_5 g(X, Z)[2(n - 1) \{(1 - k)g(W, \varphi Y) + g(W, \varphi h Y)\} \xi + [2(1 - n) + 2nk] \{-g(Y, \varphi h W) - g(Y, \varphi h W)\} \xi] \\
+ a_6 g(X, Y)[2(n - 1) \{(1 - k)g(W, \varphi Z) + g(W, \varphi h Z)\} \xi + [2(1 - n) + 2nk] \{-g(Z, \varphi W) - g(Z, \varphi h W)\} \xi] \\
+ a_7 (\nabla_W r)[g(Y, Z)X - g(X, Z)Y].
\]

Applying \( \varphi^2 \) on both sides of the above equation, we have

\[
\varphi^2((\nabla_W \tau)(X, Y)Z) = a_0 \varphi^2((\nabla_W R)(X, Y)Z) + a_7 (\nabla_W r)[g(Y, Z)\varphi^2 X - g(X, Z)\varphi^2 Y].
\]

Hence we state the following

**Theorem 3.** Let \( M \) be an \( N(k) \)-contact metric manifold. If any two of the following statements holds, then the remaining statement holds

1) \( M \) is locally \( \tau \)-\( \varphi \)-symmetric,

2) \( M \) is locally \( \varphi \)-symmetric,

3) either \( a_7 = 0 \) or \( r \) is constant.

4 **Globally \( \tau \)-\( \varphi \)-symmetric \( N(k) \)-contact metric manifold**

In this section, we define globally \( \tau \)-\( \varphi \)-symmetric \( N(k) \)-contact metric manifold by

\[
\varphi^2((\nabla_W \tau)(X, Y)Z) = 0
\]
for all vector fields $X, Y, Z, W$, which are arbitrary vector fields. By (1) and (30), we obtain

$$-((\nabla_W \tau)(X, Y)Z) + \eta((\nabla_W \tau)(X, Y)Z)\xi = 0. \quad (31)$$

By taking an innerproduct with $U$ in (31), we have

$$-g(((\nabla_W \tau)(X, Y)Z), U) + \eta((\nabla_W \tau)(X, Y)Z)g(\xi, U) = 0. \quad (32)$$

Let $\{e_i : i = 1, 2, \ldots, 2n + 1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$, in (32) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get

$$-g(((\nabla_W \tau)(e_i, Y)Z), e_i) + \eta((\nabla_W \tau)(e_i, Y)Z)g(\xi, e_i) = 0. \quad (33)$$

By using (29) in (33) and simplifying, we get

$$-a_0((\nabla_W S)(Y, Z) - \eta((\nabla_W \tau)(X, Y)Z) + \eta((\nabla_W \tau)(X, Y)Z)\xi + \eta(Y)g(h(\varphi W + \varphi hW), Z)) + 2[1(n - 1) - 1 - k]g(W, \varphi Y)\eta(Z) - g(W, h\varphi Y)\eta(Z) - \eta(Y)g(h(\varphi W, Z), Z)) + \eta(Y)g(h(\varphi W + \varphi hW), Z)) - 2a_0\eta((\nabla_W R)(\xi, Y)Z) + a_2[2(1(n - 1) + 2nk]\eta - g(W, Z)Z) - g(h(\varphi W, Z), Y)) + 2[1(n - 1) - 1 - k]g(W, \varphi Y)\eta(Z) + a_32(1(n - 1) - 1 - k]g(h(\varphi W + \varphi hW), Y))\eta(Z) + 2[1(n - 1) + 2nk]\eta - g(h(\varphi W, Y))\eta(Z) + a_4\eta((\nabla_W R)(\xi, Y)Z) + a_6[2(1(n - 1) + 2nk]\eta - g(h(\varphi W, Z), Y)) + 2[1(n - 1) - 1 - k]g(W, \varphi Z) + g(W, h\varphi Z))] + (\nabla_W r)a_7g(Y, Z) - \eta(Y)\eta(Z)) = 0. \quad (34)$$

Putting $Z = \xi$ and using the condition

$$(\nabla_W S)(Y, \xi) = S(Y, \varphi W) = S(Y, \varphi hW) - 2nk\eta g(Y, \varphi W) - 2nk\eta g(Y, \varphi hW),$$

in (34) we obtain

$$-a_0S(Y, W) + [2nk - 2(1(n - 1))(k - 1)]a_0 + [2na_1 + a_2 + a_6]2(1(n - 1) + 2nk + 2(1(n - 1)(k - 1)[a_0 + 2na_1 + a_2] - 2na_1 + a_2 + a_6][2(1(n - 1) + 2nk]\eta(Y)\eta(W) + [[2nk - 2(1(n - 1))]a_0 + [2na_1 + a_2 + a_6][2(1(n - 1) + 2nk]g(Y, \varphi W) + 2[1(n - 1)(k - 1)]a_0 + 2na_1 + a_2] - 2(n - 1)a_6[k - 1)]g(Y, W) - \eta(Y)\eta(W)) - g(h\varphi W, Y) = 0. \quad (35)$$

Replacing $W$ by $hW$ in (35), we obtain

$$g(Y, hW) = \frac{\beta}{\eta}g(Y, W) - \eta(Y)\eta(W). \quad (36)$$

where $E = [2nk - 2(n - 1)]a_0 + [2na_1 + a_2 + a_6][2(1(n - 1) + 2nk] + 2(n - 1)(a_0 + 2na_1 + a_2 + a_6)(k - 1)$ and $F = [2nk - 2(1(n - 1))]a_0 - 2(n - 1)(k - 1)(a_0 + 2na_1 + a_2 + a_6) + [2na_1 + a_2 + a_6][2(1(n - 1) + 2nk]$. By substituting (36) in (35), we obtain

$$S(Y, W) = \alpha g(Y, W) + \beta \eta(Y)\eta(W),$$

where $\alpha = \frac{\beta}{\beta_0}$ and $\beta = \frac{\beta_0 - \frac{\beta}{\alpha_0}}{\alpha_0}$. 

$$N = [2nk - 2(n - 1)]a_0 + [2na_1 + a_2 + a_6][2(1(n - 1) + 2nk] + 2(n - 1)(a_0 + 2na_1 + a_2 + a_6)(k - 1) + F = [2nk - 2(1(n - 1))]a_0 - 2(n - 1)(k - 1)(a_0 + 2na_1 + a_2 + a_6) + [2na_1 + a_2 + a_6][2(1(n - 1) + 2nk].$$

$$P = [2(n - 1)(k - 1)]a_0 + 2na_1 + a_2 - [2na_1 + a_2 + a_6][2(1(n - 1) + 2nk].$$

**Theorem 4.** A globally $\tau$-$\varphi$-symmetric $N(k)$-contact metric manifold is an $\eta$-Einstein manifold with $a_0 \neq 0$. 

ON $\varphi$-SYMMETRIC $\tau$-CURVATURE TENSOR IN $N(k)$-CONTACT METRIC MANIFOLD

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Гурунпадавва Инглахаллі, Баґеваді Ц.С. Про $\varphi$-симетричний тензор $\tau$-кривини в $N(k)$-контактному метричному многовиді // Карпатські матем. публ. — 2014. — Т.6, №2. — С. 203–211.

В цій статті вивчається тензор $\tau$-кривини в $N(k)$-контактному метричному многовиді. Досліджується $\tau$-$\varphi$-рекурентний, $\tau$-$\varphi$-симетричний і глобально $\tau$-$\varphi$-симетричний $N(k)$-контактний метричний многовид.

Ключові слова і фрази: контактний метричний многовид, симетрія.

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В этой статье изучается тензор $\tau$-кривизны в $N(k)$-контактном метрическом многообразии. Исследуется $\tau$-$\varphi$-rekurentное, $\tau$-$\varphi$-симметрическое и глобально $\tau$-$\varphi$-симметрическое $N(k)$-контактное метрическое многообразие.

Ключевые слова и фразы: контактное метрическое многообразие, симметрия.