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INDEX OF PSEUDO-PROJECTIVELY SYMMETRIC SEMI-RIEMANNIAN MANIFOLDS

The index of $\widetilde{\nabla}$-pseudo-projectively symmetric and in particular for $\widetilde{\nabla}$-projectively symmetric semi-Riemannian manifolds, where $\widetilde{\nabla}$ is Ricci symmetric metric connection are discussed.

Key words and phrases: metric connection, pseudo-projective curvature tensor, projective curvature tensor, semi-Riemannian manifold, index of a manifold.

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INTRODUCTION

In 1923, Eisenhart [2] obtained the condition for the existence of a second order parallel symmetric tensor in a Riemannian manifold and proved that if a Riemannian manifold admits a second order parallel symmetric tensor other than a constant multiple of the Riemannian metric, then it is reducible. In 1925, Levy [9] gave the necessary and sufficient condition for the existence of second order parallel symmetric tensors and proved that a second order parallel symmetric non-singular tensor in a real space form is always proportional to the Riemannian metric. After that Sharma [13] improved the result of Levy and proved that any second order parallel tensor (not necessarily symmetric) in a real space form of dimension greater than 2 is proportional to the Riemannian metric. Later in 1939, Thomas [17] defined and studied the index of a Riemannian manifold. A set of metric tensors (i.e. symmetric non-degenerate parallel $(0, 2)$ tensor field on the differentiable manifold) $\{H_1, \ldots, H_\ell\}$ is said to be *linearly independent* if

$$c_1 H_1 + \cdots + c_\ell H_\ell = 0, \qquad c_1, \ldots, c_\ell \in \mathbf{R},$$

implies that $c_1 = \cdots = c_\ell = 0$.

The set of metric tensors $\{H_1, \ldots, H_\ell\}$ is said to be a complete set if any metric tensor H can be written as

$$H = c_1 H_1 + \cdots + c_\ell H_\ell, \qquad c_1, \ldots, c_\ell \in \mathbf{R}.$$

More precisely, the number of linearly independent metric tensors in a complete set of metric tensors of a Riemannian manifold is called the index of the Riemannian manifold [17, p. 413]. Therefore the existence of a second order parallel symmetric tensor is very closely related with the index of Riemannian manifolds. Then in 1968, Levine and Katzin [8] proved that the index of an n-dimensional conformally flat manifold is $n(n+1)/2$ or 1 according as it is a

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flat manifold or a manifold of non-zero constant curvature. In 1981, Stavre [14] proved that if the index of an n-dimensional conformally symmetric Riemannian manifold (except the four cases of being conformally flat, of constant curvature, an Einstein manifold or with covariant constant Einstein tensor) is greater than 1, then it must be between 2 and $n+1$. In 1982, Starve and Smaranda [16] found the index of a conformally symmetric Riemannian manifolds with respect to a semi-symmetric metric connection of Yano [22]. In the recent paper [18] author and Tripathi studied the index of quasi-conformally symmetric, conformally symmetric and concircularly symmetric semi-Riemannian manifolds with respect to any metric connection and discussed some applications.

The index of the conformally flat and conformally symmetric (with respect to the Levi-Civita connection, semi-symmetric metric connection of Yano [22] and metric connection) (semi-)Riemannian manifolds were studied by many authors [8, 14, 16, 18]. Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view and the pseudo-projective curvature tensor is a generalized case of projective curvature tensor. A real space form is always pseudo-projectively flat and a pseudo-projectively flat manifold is always pseudo-projectively symmetric. But the converse is not true in both cases. The study of manifolds with semi-Riemannian metrics is of interest from the stand point of physics and relativity and have been studied by several authors. Motivated by these studies, in this paper we study the index of pseudo-projectively symmetric and projectively symmetric semi-Riemannian manifolds with respect to the metric connection $\widetilde{\nabla}$. The paper is organized as follows: In Section 1, we give the preliminaries about the index of a semi-Riemannian manifold and Ricci-symmetric metric connection. In Section 2, the definition of the pseudo-projective curvature tensor in terms of projective curvature tensor and concircular curvature tensor with respect to a metric connection $\widetilde{\nabla}$ are given. We also obtain a complete classification of $\widetilde{\nabla}$-pseudo-projective flat (in particular, pseudo-projective flat) manifolds. In Section 3, we find out the index of $\widetilde{\nabla}$-pseudo-projectively symmetric and $\widetilde{\nabla}$-projectively symmetric semi-Riemannian manifolds. In the last section, some applications in theory of relativity are discussed.

1 Preliminaries

Let M be an n-dimensional differentiable manifold. Let $\widetilde{\nabla}$ be a linear connection in M. Then torsion tensor $\widetilde{T}$ and curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ are given by

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X, \quad \widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z.$$

By a semi-Riemannian metric [10] on M, we understand a non-degenerate symmetric $(0,2)$ tensor field g. In [17], a semi-Riemannian metric is called a metric tensor, a positive definite symmetric $(0,2)$ tensor field, that is, Riemannian metric is called a fundamental metric tensor and a symmetric $(0,2)$ tensor field g of rank less than n is called a degenerate metric tensor.

Let (M,g) be an n-dimensional semi-Riemannian manifold. A linear connection $\widetilde{\nabla}$ in M is called a metric connection with respect to the semi-Riemannian metric g if $\widetilde{\nabla} g = 0$. If the torsion tensor of the metric connection $\widetilde{\nabla}$ is zero, then it becomes Levi-Civita connection ∇, which is unique by the fundamental theorem of Riemannian geometry. If the torsion tensor of the metric connection $\widetilde{\nabla}$ is not zero, then it is called a Hayden connection [6, 23]. Semi-symmetric metric connections [22] and quarter symmetric metric connections [4] are some well known examples of Hayden connections.

For a metric connection $\widetilde{\nabla}$ in an n-dimensional semi-Riemannian manifold (M,g), the curvature tensor $\widetilde{R}$ with respect to the $\widetilde{\nabla}$ satisfies the following conditions

$$\widetilde{R}\,(X,Y,Z,V)+\widetilde{R}\,(Y,X,Z,V)=0, \tag{1}$$

$$\widetilde{R}\,(X,Y,Z,V)+\widetilde{R}\,(X,Y,V,Z)=0, \tag{2}$$

where

$$\widetilde{R}\,(X,Y,Z,V)=g(\widetilde{R}\,(X,Y)\,Z,V).$$

Let $\{e_1,\dots,e_n\}$ be any orthonormal basis of vector fields in the manifold M. The Ricci tensor $\widetilde{S}$ and the scalar curvature $\widetilde{r}$ of the semi-Riemannian manifold with respect to the metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{S}\,(X,Y)=\sum_{i=1}^{n}\widetilde{R}\,(e_i,X,Y,e_i)\,,\widetilde{r}=\sum_{i=1}^{n}\widetilde{S}\,(e_i,e_i)\,.$$

The Ricci operator $\widetilde{Q}$ with respect to the metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{S}\,(X,Y)=g(\widetilde{Q}X,Y).$$

Define

$$\widetilde{e}X=\widetilde{Q}X-\frac{\widetilde{r}}{n}X$$

and

$$\widetilde{E}\,(X,Y)=g\,(\widetilde{e}X,Y)\,.$$

Then

$$\widetilde{E}=\widetilde{S}-\frac{\widetilde{r}}{n}\,g.$$

The $(0,2)$ tensor $\widetilde{E}$ is known as tensor of Einstein [15] with respect to the metric connection $\widetilde{\nabla}$. $\widetilde{S}$ is symmetric if and only if $\widetilde{E}$ is symmetric.

Definition 1 ([18]). *A metric connection $\widetilde{\nabla}$ with symmetric Ricci tensor $\widetilde{S}$ is called a Ricci-symmetric metric connection.*

For more details about Ricci-symmetric metric connection see [18].

Definition 2 ([18]). *Let (M,g) be an n-dimensional semi-Riemannian manifold equipped with a metric connection $\widetilde{\nabla}$. A symmetric $(0,2)$ tensor field H, which is covariantly constant with respect to $\widetilde{\nabla}$, is called a special quadratic first integral (for brevity SQFI) [7] with respect to $\widetilde{\nabla}$. The semi-Riemannian metric g is always an SQFI. A set of SQFI tensors $\{H_1,\dots,H_\ell\}$ with respect to $\widetilde{\nabla}$ is said to be linearly independent if*

$$c_1H_1+\cdots+c_\ell H_\ell=0,\qquad c_1,\dots,c_\ell\in\mathbf{R},$$

implies that $c_1=\cdots=c_\ell=0$.

The set $\{H_1,\dots,H_\ell\}$ is said to be a complete set if any SQFI tensor H with respect to $\widetilde{\nabla}$ can be written as $H=c_1H_1+\cdots+c_\ell H_\ell,\qquad c_1,\dots,c_\ell\in\mathbf{R}$.

The index [17] of the manifold M with respect to $\widetilde{\nabla}$, denoted by $i_{\widetilde{\nabla}}$, is defined as the number ℓ of members in a complete set $\{H_1,\dots,H_\ell\}$. Hence the index $i_{\widetilde{\nabla}}$ of the manifold M with respect to the metric connection $\widetilde{\nabla}$ is the maximum number of linearly independent SQFI in a complete set of SQFI.

2 Pseudo-projective curvature tensor

Let (M, g) be an n-dimensional $(n > 2)$ semi-Riemannian manifold equipped with a metric connection $\widetilde{\nabla}$. The projective curvature tensor $\widetilde{\mathcal{P}}$ with respect to the $\widetilde{\nabla}$ is defined by [3, p. 90]

$$\widetilde{\mathcal{P}}(X,Y,Z,V) = \widetilde{R}(X,Y,Z,V) - \frac{1}{n-1}(\widetilde{S}(Y,Z)\, g(X,V) - \widetilde{S}(X,Z)\, g(Y,V)), \tag{3}$$

and the concircular curvature tensor $\widetilde{\mathcal{Z}}$ with respect to $\widetilde{\nabla}$ is defined by ([21], [24, p. 87])

$$\widetilde{\mathcal{Z}}(X,Y,Z,V) = \widetilde{R}(X,Y,Z,V) - \frac{\widetilde{r}}{n(n-1)}(g(Y,Z)\, g(X,V) - g(X,Z)\, g(Y,V)). \tag{4}$$

As a generalization of the notion of projective curvature tensor and concircular curvature tensor, the pseudo-projective curvature tensor $\widetilde{\mathcal{P}}_*$ with respect to $\widetilde{\nabla}$ is defined by [12]

$$\begin{aligned} \widetilde{\mathcal{P}}_*(X,Y,Z,V) &= a\widetilde{R}(X,Y,Z,V) \\ &\quad + b\left(\widetilde{S}(Y,Z)\, g(X,V) - \widetilde{S}(X,Z)\, g(Y,V)\right) \\ &\quad - \frac{\widetilde{r}}{n}\left(\frac{a}{n-1} + b\right)(g(Y,Z)\, g(X,V) - g(X,Z)\, g(Y,V)), \end{aligned} \tag{5}$$

where a and b are constants. In fact, we have

$$\widetilde{\mathcal{P}}_*(X,Y,Z,V) = -(n-1)b\,\widetilde{\mathcal{P}}(X,Y,Z,V) + (a+(n-1)b)\widetilde{\mathcal{Z}}(X,Y,Z,V).$$

Since, there is no restrictions for manifolds if $a = 0$ and $b = 0$, therefore it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. From (5) it is clear that if $a = 1$ and $b = -1/(n-1)$, then $\widetilde{\mathcal{P}}_* = \widetilde{\mathcal{P}}$; and if $a = 1$ and $b = 0$, then $\widetilde{\mathcal{P}}_* = \widetilde{\mathcal{Z}}$.

Now, we need the following

Definition 3. *A semi-Riemannian manifold (M, g) equipped with a metric connection $\widetilde{\nabla}$ is said to be:*

(a) *$\widetilde{\nabla}$-pseudo-projectively flat if $\widetilde{\mathcal{P}}_* = 0$;*

(b) *$\widetilde{\nabla}$-projectively flat if $\widetilde{\mathcal{P}} = 0$;*

(c) *$\widetilde{\nabla}$-concircularly flat if $\widetilde{\mathcal{Z}} = 0$.*

In particular, with respect to the Levi-Civita connection ∇, $\widetilde{\nabla}$-pseudo-projectively flat, $\widetilde{\nabla}$-projectively flat and $\widetilde{\nabla}$-concircularly flat become simply pseudo-projectively flat, projectively flat and concircularly flat respectively.

Definition 4. *A semi-Riemannian manifold (M, g) equipped with a metric connection $\widetilde{\nabla}$ is said to be:*

(a) *$\widetilde{\nabla}$-pseudo-projectively symmetric if $\widetilde{\nabla}\widetilde{\mathcal{P}}_* = 0$;*

(b) *$\widetilde{\nabla}$-projectively symmetric if $\widetilde{\nabla}\widetilde{\mathcal{P}} = 0$;*

(c) *$\widetilde{\nabla}$-concircularly symmetric if $\widetilde{\nabla}\widetilde{\mathcal{Z}} = 0$.*

In particular, with respect to the Levi-Civita connection ∇, $\widetilde{\nabla}$-pseudo-projectively symmetric, $\widetilde{\nabla}$-projectively symmetric and $\widetilde{\nabla}$-concircularly symmetric become simply pseudo-projectively symmetric, projectively symmetric and concircularly symmetric respectively.

Theorem 1. *Let M be a semi-Riemannian manifold of dimension n greater than 2. Then M is $\widetilde{\nabla}$-pseudo-projectively flat if and only if one of the following statement is true:*

(i) $a+(n-1)b=0$, $a\neq 0\neq b$ *and M is $\widetilde{\nabla}$-projectively flat;*

(ii) $a+(n-1)b\neq 0$, $a\neq 0$, *M is $\widetilde{\nabla}$-projectively flat and $\widetilde{\nabla}$-concircularly flat;*

(iii) $a+(n-1)b\neq 0$, $a=0$ *and Ricci tensor $\widetilde{S}$ with respect to $\widetilde{\nabla}$ satisfies*

$$\widetilde{S}-\frac{\widetilde{r}}{n}g=0, \tag{6}$$

where $\widetilde{r}$ is the scalar curvature with respect to $\widetilde{\nabla}$.

Proof. Using $\widetilde{\mathcal{P}}_*=0$ in (5) we get

$$\begin{aligned}0=&a\widetilde{R}(X,Y,Z,V)+b(\widetilde{S}(Y,Z)g(X,V)-\widetilde{S}(X,Z)g(Y,V))\\&-\frac{\widetilde{r}}{n}\left(\frac{a}{n-1}+b\right)(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)),\end{aligned} \tag{7}$$

from which we obtain

$$(a+(n-1)b)\left(\widetilde{S}-\frac{\widetilde{r}}{n}g\right)=0. \tag{8}$$

Case 1. $a+(n-1)b=0$ and $a\neq 0\neq b$. Then from (5) and (3), it follows that $(n-1)b\widetilde{\mathcal{P}}=0$, which gives $\widetilde{\mathcal{P}}=0$. This gives the statement (i).

Case 2. $a+(n-1)b\neq 0$ and $a\neq 0$. Then from (8), we have

$$\widetilde{S}(Y,Z)=\frac{\widetilde{r}}{n}g(Y,Z). \tag{9}$$

Using (9) in (7), we get

$$a(\widetilde{R}(X,Y,Z,V)-\frac{\widetilde{r}}{n(n-1)}(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)))=0. \tag{10}$$

Since $a\neq 0$, then by (4), we get $\widetilde{\mathcal{Z}}=0$ and by using (10), (9) in (3), we get $\widetilde{\mathcal{P}}=0$. This gives the statement (ii).

Case 3. $a+(n-2)b\neq 0$ and $a=0$, we get (6). This gives the statement (iii). Converse is true in all cases. □

Corollary 1. [19] *Let M be a semi-Riemannian manifold of dimension n greater than 2. Then M is pseudo-projectively flat if and only if one of the following statement is true:*

(i) $a+(n-1)b=0$, $a\neq 0\neq b$ *and M is projectively flat;*

(ii) $a+(n-1)b\neq 0$, $a\neq 0$, *M is real space form;*

(iii) $a+(n-1)b\neq 0$, $a=0$ *and M is Einstein manifold.*

3 INDEX OF PSEUDO-PROJECTIVE SYMMETRIC MANIFOLDS

Let (M,g) be an n-dimensional semi-Riemannian manifold equipped with the metric connection $\widetilde{\nabla}$ and $\widetilde{R}$ be the curvature tensor of M with respect to the metric connection $\widetilde{\nabla}$. The integrability condition for the SQFI H is given by

$$H((\widetilde{\nabla}_U\widetilde{R})(X,Y)Z,V)+H(Z,(\widetilde{\nabla}_U\widetilde{R})(X,Y)V)=0. \tag{11}$$

Therefore, the solutions H of (11) is closely related to the index of pseudo-projectively symmetric and projectively symmetric semi-Riemannian manifolds with respect to the $\widetilde{\nabla}$.

Lemma 1. *If (M,g) be an n-dimensional semi-Riemannian $\widetilde{\nabla}$-pseudo-projectively symmetric manifold and $n>2$, $b\neq 0$. Then*

$$\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})=0,$$

where U is an arbitrary vector field.

Proof. Using (1) in (5), we get

$$\begin{aligned}\widetilde{\mathcal{P}}_*(X,Y,Z,V)&=a\widetilde{R}(X,Y,Z,V)+b(\widetilde{E}(Y,Z)g(X,V)-\widetilde{E}(X,Z)g(Y,V))\\&\quad-\frac{a\widetilde{r}}{n(n-1)}(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)).\end{aligned}\tag{12}$$

Taking the covariant derivative of (12) and using $\widetilde{\nabla}_U\widetilde{\mathcal{P}}_*=0$, we get

$$\begin{aligned}-a(\widetilde{\nabla}_U\widetilde{R})(X,Y,Z,V)&=b\Big((\widetilde{\nabla}_U\widetilde{E})(Y,Z)g(X,V)-(\widetilde{\nabla}_U\widetilde{E})(X,Z)g(Y,V)\Big)\\&\quad-\frac{(\widetilde{\nabla}_U\widetilde{r})a}{n(n-1)}(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)).\end{aligned}\tag{13}$$

Contracting Y and Z in (13) and using the condition (1) and (2), we have

$$\begin{aligned}-a(\widetilde{\nabla}_U\widetilde{S})(X,V)&=b\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})g(X,V)-(\widetilde{\nabla}_U\widetilde{E})(X,V)\\&\quad-\frac{(\widetilde{\nabla}_U\widetilde{r})a}{n}g(X,V).\end{aligned}\tag{14}$$

Taking $X=V=e_j$ in (14), we obtain

$$\begin{aligned}&b(n-1)\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})=0,\\&\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})=0,\qquad(\text{since } b\neq 0 \text{ and } n>2).\end{aligned}\tag{15}$$

□

Theorem 2. *Let (M,g) be an n-dimensional semi-Riemannian $\widetilde{\nabla}$-pseudo-projective symmetric manifold with $n>2$ and $b\neq 0$, then the equation (11) has maximum number of solution and consequently, $i_{\widetilde{\nabla}}=\frac{1}{2}n(n-1)$.*

Proof. Using (13) and (11), we find

$$\begin{aligned}0&=b((\widetilde{\nabla}_U\widetilde{E})(Y,Z)H(X,V)-(\widetilde{\nabla}_U\widetilde{E})(X,Z)H(Y,V)\\&\quad+(\widetilde{\nabla}_U\widetilde{E})(Y,V)H(X,Z)-(\widetilde{\nabla}_U\widetilde{E})(X,V)H(Y,Z))\\&\quad-\frac{a(\widetilde{\nabla}_U\widetilde{r})}{n(n-1)}(g(Z,Y)H(X,V)-g(Z,X)H(Y,V)\\&\quad+g(V,Y)H(X,Z)-g(V,X)H(Y,Z)).\end{aligned}\tag{16}$$

Taking $X = Z = e_i$ in (16) and using (15), we get

$$\begin{aligned} &b(H((\widetilde{\nabla}_U\widetilde{e})Y,V) - H((\widetilde{\nabla}_U\widetilde{e})V,Y) + (\widetilde{\nabla}_U\widetilde{E})\,(V,Y)\,\mathrm{trace}(H)) \\ &= \frac{a(\widetilde{\nabla}_U\widetilde{r})}{n(n-1)}(-nH\,(Y,V) + g\,(Y,V)\,\mathrm{trace}(H)). \end{aligned} \tag{17}$$

Interchanging V with Y in (17) and then subtracting the resulting equation from (17), we obtain

$$H((\widetilde{\nabla}_U\widetilde{e})Y,V) = H((\widetilde{\nabla}_U\widetilde{e})V,Y). \tag{18}$$

Using (18) in (17), we get

$$b(\widetilde{\nabla}_U\widetilde{E})\,(V,Y) = \frac{a(\widetilde{\nabla}_U\widetilde{r})}{n(n-1)}(g\,(Y,V) - \frac{n}{\mathrm{trace}(H)}H(Y,V)). \tag{19}$$

Now, interchanging X with Z, and Y with V in (16) and taking the sum of the resulting equation and (16) and using (19), we see that the equation (11) is satisfied identically. Thus the equation has the maximum number of solutions for a $\widetilde{\nabla}$-pseudo-projective symmetric semi-Riemannian manifold. Consequently, M admits the maximum number of linearly independent SQFI. So, the index of a $\widetilde{\nabla}$ -pseudo-projectively symmetric semi-Riemannian manifold is

$$i_{\widetilde{\nabla}} = \frac{1}{2}n\,(n-1)\,.$$

□

Corollary 2. *If (M,g) is an n-dimensional semi-Riemannian $\widetilde{\nabla}$-projectively symmetric manifold, then the equation* (11) *has maximum number of solution and consequently, $i_{\widetilde{\nabla}} = \frac{1}{2}n(n-1)$.*

4 CONCLUSION

A semi-Riemannian manifold is said to be *decomposable* [17] (or locally reducible) if there always exists a local coordinate system (x^i) so that its metric takes the form

$$ds^2 = \sum_{a,b=1}^{r} g_{ab}dx^a dx^b + \sum_{\alpha,\beta=r+1}^{n} g_{\alpha\beta}dx^\alpha dx^\beta,$$

where g_{ab} are functions of $x^1,\ldots,x^r$ and $g_{\alpha\beta}$ are functions of $x^{r+1},\ldots,x^n$. A semi-Riemannian manifold is said to be *reducible* if it is isometric to the product of two or more semi-Riemannian manifolds; otherwise it is said to be *irreducible* [17]. A reducible semi-Riemannian manifold is always decomposable but the converse need not be true.

The concept of the index of a (semi-)Riemannian manifold gives a striking tool to decide the reducibility and decomposability of (semi-)Riemannian manifolds. For example, a Riemannian manifold is decomposable if and only if its index is greater than one [17]. Moreover, a complete Riemannian manifold is reducible if and only if its index is greater than one [17]. A second order $(0,2)$-symmetric parallel tensor is also known as a special Killing tensor of order two. Thus, a Riemannian manifold admits a special Killing tensor other than the Riemannian metric g if and only if the manifold is reducible [2], that is the index of the manifold is greater than 1. In 1951, Patterson [11] found a similar result for semi-Riemannian manifolds. In fact, he proved that a semi-Riemannian manifold (M, g) admitting a special Killing tensor K_{ij}, other than g, is reducible if the matrix (K_{ij}) has at least two distinct characteristic roots at every point of the manifold. In this case, the index of the manifold is again greater than 1.

By Theorem 2, we conclude that a $\widetilde{\nabla}$-pseudo-projectively symmetric Riemannian manifold (where $\widetilde{\nabla}$ is any Ricci symmetric metric connection, not necessarily Levi-Civita connection) is decomposable and it is reducible if the manifold is complete.

It is known that the maximum number of linearly independent Killing tensors of order 2 in a semi-Riemannian manifold (M^n, g) is $\frac{1}{12}n(n+1)^2(n+2)$, which is attained if and only if M is of constant curvature. The space of constant curvature and projectively flat space are identical classes. Therefore the maximum number of linearly independent Killing tensors of order 2 in a semi-Riemannian manifold (M^n, g) is $\frac{1}{12}n(n+1)^2(n+2)$, which is attained if and only if M is projectively flat. The maximum number of linearly independent Killing tensors in a 4-dimensional spacetime is 50 and this number is attained if and only if the spacetime is of constant curvature [5] or projectively flat. But spaces of constant curvature do not admit special quadratic first integrals. From Theorem 2, we also conclude that the maximum number of linearly independent special Killing tensors, that is, SQFI in a 4-dimensional spacetime is 6.

From the physical point of view Killing tensors are important because they provide quadratic first integrals of the geodesics. It is shown that [1] the special quadratic first integrals can be written as the sum of products of two linear first integrals only if the space admits a covariantly constant vector. Therefore special quadratic first integrals are useful in the analysis of the geodesics of given relativistic space-times possessing groups of motion of order less than or equal to 2.

The charged Kerr solution with or without cosmological constant admits a quadratic first integral which is irreducible provided the angular momentum parameter is not zero [20]. But this quadratic first integral is not special [1].

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Досліджується індекс $\widetilde{\nabla}$-псевдопроективно симетричних і зокрема $\widetilde{\nabla}$-проективно симетричних напіврімановних многовидів, де $\widetilde{\nabla}$ — це симетричний метричний зв'язок Річчі.

Ключові слова і фрази: метричний зв'язок, псевдопроективний тензор кривизни, проективний тензор кривизни, напіврімановий многовид, індекс многовиду.