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A CLASS OF JULIA EXCEPTIONAL FUNCTIONS

The class of \( p \)-loxodromic functions (meromorphic functions, satisfying the condition \( f(qz) = pf(z) \) for some \( q \in \mathbb{C}\setminus\{0\} \) and all \( z \in \mathbb{C}\setminus\{0\} \)) is studied. Each \( p \)-loxodromic function is Julia exceptional. The representation of these functions as well as their zero and pole distribution are investigated.

Key words and phrases: \( p \)-loxodromic function, the Schottky-Klein prime function, Julia exceptionality.

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INTRODUCTION

Denote \( \mathbb{C}^* = \mathbb{C}\setminus\{0\} \), and let \( q, p \in \mathbb{C}^*, |q| < 1 \).

Definition 1. A meromorphic in \( \mathbb{C}^* \) function \( f \) is said to be \( p \)-loxodromic of multiplicator \( q \) if for every \( z \in \mathbb{C}^* \)

\[
f(qz) = pf(z).
\]

Let \( L_{qp} \) denotes the class of \( p \)-loxodromic functions of multiplicator \( q \).

The case \( p = 1 \) has been studied earlier in the works of O. Rausenberger [9], G. Valiron [11] and Y. Hellegouarch [5]. In his work [3, p. 133] which A. Ostrowski [8] called "besonders schöne und überraschende" G. Julia gave an example of a meromorphic in the punctured plane \( \mathbb{C}^* \) function satisfying (1) with \( p = 1 \) for some non-zero \( q, |q| \neq 1 \), and all \( z \in \mathbb{C}^* \). He noted that the family \( \{f_n(z)\}, f_n(z) = f(q^n z) \) is normal [7] in \( \mathbb{C}^* \) because \( f_n(z) = f(z) \) for all \( z \in \mathbb{C}^* \).

If \( p = 1 \) the function \( f \) is called loxodromic. Loxodromic functions of multiplicator \( q \) form a field, which is denoted by \( L_q \). The set \( L_{qp} \) forms an Abelian group with respect to addition. It is obvious that a ratio of two functions from \( L_{qp} \) is a loxodromic function, and the derivative of the loxodromic function is \( p \)-loxodromic with \( p = \frac{1}{q} \).

Remark 1. Every \( f \equiv \text{const} \) belongs to \( L_q \), but the unique constant function belonging to \( L_{qp} \) is \( f \equiv 0 \).

If \( f \in L_{qp} \) and \( a \) is a zero of \( f \), then \( aq^n, n \in \mathbb{Z} \), are as well. That is, in the case of non-positive \( q \) the zeros of \( f \) lay on a logarithmic spiral. Let \( a = |a|e^{i\alpha}, q = |q|e^{i\gamma} \). Then the logarithmic spiral in polar coordinates \((r, \varphi)\) takes the form

\[
\log r - \log |a| = k(\varphi - \alpha),
\]
where \( k = \frac{\log |q|}{\gamma} \). The same concerns the poles of \( f \). The image of a logarithmic spiral on the Riemann sphere by the stereographic projection intersects each meridian at the same angle and is called loxodromic curve (\( \lambda \odot \delta \rho \odot \zeta \) - oblique, \( \delta \rho \odot \mu \odot \zeta \) - way). That is why we call (following G. Valiron) the function from \( L_q \) loxodromic.

Remark 2. If \( f \in L_q \) and \( z \) is its a-point, \( a \in \mathbb{C} \cup \{ \infty \} \), then \( q^n z, n \in \mathbb{Z} \), are its a-points too. In the case, \( f \in L_{qp} \), the previous considerations are valid only for the zeros and the poles of \( f \).

It is easy to verify, that \( L_{qp} \) forms the linear spaces over the fields \( \mathbb{C} \) and \( L_q \). Also it is clear that \( L_{qp} \) has the following properties.

**Proposition.** The linear space \( L_{qp} \) has the following properties.
1. The map \( D : f(z) \mapsto zf'(z) \) maps \( L_{qp} \) to \( L_{qp} \).
2. The map \( D_1 : f(z) \mapsto z\frac{f'(z)}{f(z)} \) maps \( L_{qp} \) to \( L_q \).
3. \( f(z) \in L_{qp} \Rightarrow f\left(\frac{1}{z}\right) \in L_{qp} \).

Let us give nontrivial example of \( p \)-loxodromic function of multiplicator \( q \). Put
\[
h(z) = \prod_{n=1}^{\infty} (1-q^n z), \quad 0 < |q| < 1.
\]

**Definition 2.** The function
\[
P(z) = (1-z)h(z)h\left(\frac{1}{z}\right) = (1-z)\prod_{n=1}^{\infty} (1-q^n z)(1-\frac{q^n}{z})
\]
is called the Schottky-Klein prime function.

This function is holomorphic in \( \mathbb{C}^* \) with zero sequence \( \{q^n\}, n \in \mathbb{Z} \). It was introduced by Schottky [10] and Klein [6] for the study of conformal mappings of doubly-connected domains, see also [2].

It is easy to obtain the following property of \( P \)
\[
P(qz) = -\frac{1}{z}P(z).
\]

**Example 1.** Consider the function
\[
f(z) = \frac{P\left(\frac{z}{P}\right)}{P(z)}.
\]

Using (2), it is easy to show that \( f \in L_{qp} \).

1 **The Numbers of Zeros and Poles of \( p \)-L o xodromic Functions in an Annulus**

Let \( A_q(R) = \{z \in \mathbb{C} : |q|R < |z| \leq R\} \), \( R > 0 \) and \( A_q = A_q(1) \).

**Theorem 1.** Let \( f \in L_{qp} \) and the boundary of \( A_q(R) \) contains neither zeros nor poles of \( f \). Then \( f \) has equal numbers of zeros and poles (counted according to their multiplicities) in every \( A_q(R) \).
Proof. Let $\Gamma_1 = \{ z \in \mathbb{C} : |z| = |qR| \}$ and $\Gamma_2 = \{ z \in \mathbb{C} : |z| = R \}$ denote the circles bounding $A_q(R)$. Let $n(f)$ be the number of poles of $f$ in $A_q(R)$.

By the argument principle, we have

$$ n\left(\frac{1}{f}\right) - n(f) = \frac{1}{2i\pi} \left( \int_{\Gamma_2^+} \frac{f'(z)}{f(z)} dz - \int_{\Gamma_1^+} \frac{f'(\xi)}{f(\xi)} d\xi \right). \quad (3) $$

Setting $\xi = qz$ in the second integral of (3), we obtain

$$ n\left(\frac{1}{f}\right) - n(f) = \frac{1}{2i\pi} \int_{\Gamma_2^+} \left( \frac{f'(z)}{f(z)} - q\frac{f'(qz)}{f(qz)} \right) dz. \quad (4) $$

Since $f \in L_q p$, the relation (1) implies

$$ f'(qz) = \frac{p}{q} f'(z). \quad (5) $$

Putting (1) and (5) in (4), we obtain the conclusion of the theorem. \(\square\)

**Remark 3.** Every non-constant loxodromic function of multiplicator $q$ has at least two poles (and two zeros) in every annulus $A_q(R)$ [5]. As we see from Example 1, the $p$-loxodromic function $f$ has the unique pole $z = 1$ in $A_q$. This is an essential difference between loxodromic and $p$-loxodromic functions with $p \neq 1$.

2 \ REPRESENTATION OF $p$-LOXODROMIC FUNCTIONS

The representation of loxodromic functions from $L_q$ was given in [11], [5]. The following theorem gives the representation of a function from $L_{qp}$.

Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ be the zeros and the poles of $f \in L_{qp}$ in $A_q$ respectively. Denote

$$ \lambda = \frac{a_1 \cdots a_m}{b_1 \cdots b_m}. \quad (6) $$

**Theorem 2.** The non-identical zero meromorphic in $\mathbb{C}^*$ function $f$ belongs to $L_{qp}$, $p \neq 1$, if and only if there exists $v \in \mathbb{Z}$ such that $\lambda = \frac{p^v}{q^v}$ and $f$ has the form

$$ f(z) = c z^v \frac{P\left(\frac{z}{a_1}\right) \cdots P\left(\frac{z}{a_m}\right)}{P\left(\frac{z}{b_1}\right) \cdots P\left(\frac{z}{b_m}\right)}, \quad (7) $$

where $c$ is a constant.

**Proof.** Firstly, denote

$$ M(z) = \frac{P\left(\frac{z}{a_1}\right) \cdots P\left(\frac{z}{a_m}\right)}{P\left(\frac{z}{b_1}\right) \cdots P\left(\frac{z}{b_m}\right)} $$

and consider the function

$$ g(z) = \frac{f(z)}{M(z)}. $$
Since the functions $f$ and $M$ have the same zeros and poles, it follows that their ratio $g$ is holomorphic in $\mathbb{C}^*$ function. Let $g(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$ be the Launart expansion of $g$ in $\mathbb{C}^*$. Using relation (1) and the equality (2), we obtain
\[ \lambda g(qz) = pg(z). \] (8)

According to (8), we obtain
\[ \lambda \sum_{n=-\infty}^{+\infty} c_n q^n z^n = p \sum_{n=-\infty}^{+\infty} c_n z^n \]
for any $z \in \mathbb{C}^*$. This implies $\lambda c_n q^n = pc_n$ or $c_n (\lambda q^n - p) = 0$. Then there exists at least one $c_n \neq 0$, $n \in \mathbb{Z}$, such that
\[ c_n (\lambda q^n - p) = 0. \] (9)

Hence, the relation (9) implies $q^n = \frac{p}{\lambda}$. We see also that $c_n = 0$ if $n \neq n$, so we have $g(z) = c_n z^n$. Thus, we can conclude
\[ f(z) = g(z) M(z) = cz^n M(z), \]
where $c$ is a constant.

Secondly, we have $f(z) = cz^n M(z)$, $n \in \mathbb{Z}$. Show that it belongs to $\mathcal{L}_{qp}$. Thus, $f(qz) = cq^n z^n M(qz)$. Indeed, using (2), we obtain
\[ f(qz) = cq^n z^n \lambda M(z) = pf(z). \]

This completes the proof.

**Corollary 1.** Assume $f \in \mathcal{L}_{qp}$, if $f$ is holomorphic in $\mathbb{C}^*$, then $f(z) \equiv 0$ or there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $p = q^k$ and $f(z) = cz^k$, where $c$ is a constant. Conversely, a holomorphic in $\mathbb{C}^*$ function of the form $f(z) = cz^k$, where $k \in \mathbb{Z} \setminus \{0\}$, $c$ is a constant, belongs to $\mathcal{L}_{qp}$.

### 3 Zero and pole distribution

Let $\{a_j\}, \{b_j\}, j \in \mathbb{Z}$ be a couple of sequences in $\mathbb{C}^*$, $p \neq 1$. Put
\[ \mu(r) = \log r / \log |q| - 1. \]

Note that $\mu(r) = 0$ if $|q| \leq r < 1$. Denote
\[
\mathcal{M}(\nu)(r) = \frac{1}{|p|^{\mu(r)}} \times \begin{cases} 
    r^\nu \frac{\prod_{1<|a_j| \leq r} |a_j|}{\prod_{1<|b_j| \leq r} |b_j|}, & r > 1; \\
    \frac{\prod_{r<|a_j| \leq 1} |a_j|}{\prod_{r<|b_j| \leq 1} |b_j|}, & 0 < r \leq 1.
\end{cases}
\]
Theorem 3. The zero sequence \( \{a_j\} \) and the pole sequence \( \{b_j\} \) of a non-identical zero meromorphic \( p \)-loxodromic function of multiplicator \( q \) satisfy the following conditions:

(i) the number of \( a_j \) and \( b_j \) in every annulus of the form \( \{z : r < |z| < 2r\}, r > 0 \) is bounded by an absolute constant;

(ii) the difference between the numbers of \( a_j \) and \( b_k \) in every annulus \( \{z : r_1 < |z| < r_2\}, 0 < r_1 < r_2 < +\infty \) is bounded by an absolute constant;

(iii) there exists \( C_1 > 0 \) such that

\[
\left| \frac{a_j}{b_k} - 1 \right| > C_1 \quad \text{for every } j, k \in \mathbb{Z};
\]

(iv) the function \( \mathcal{M}_v(r) \), where \( v \in \mathbb{Z} \) such that \( \lambda = \frac{\nu}{q} \), and \( \lambda \) is given by (6), is bounded for \( r > 0 \).

Proof. Let \( f \) be a \( p \)-loxodromic of multiplicator \( q \) function. If \( f \) is holomorphic then by Corollary 1 there exists \( k \in \mathbb{Z} \setminus \{0\} \) such that \( f(z) = cz^k \), and \( c \) is a constant. Hence, \( f \) has no zeros in \( C^* \). So there is nothing to prove.

Let \( f \) be meromorphic. Then by Remark 2 and Theorem 1 it has infinitely many zeros and poles.

(i) First we remark that there exists a unique \( n_0 \in \mathbb{Z}_+ \) such that \( \frac{1}{|q|^{n_0}} \leq 2 < \frac{1}{|q|^{n_0+1}} \). This \( n_0 \) is equal to

\[
\left\lfloor \frac{\log 2}{\log |q|} \right\rfloor.
\]

Since every annulus \( \{z : \frac{r}{|q|^k} < |z| \leq \frac{r}{|q|^{k+1}}\} \), where \( k \in \mathbb{Z} \), contains the same number of zeros of \( f \), say \( m \), and

\[
(r, 2r) = \left( \bigcup_{k=0}^{n_0-1} \left( \frac{r}{|q|^k}, \frac{r}{|q|^{k+1}} \right) \right) \cup \left( \frac{r}{|q|^{n_0}} , 2r \right)
\]

it follows that the annulus \( \{z : r < |z| \leq 2r\} \) contains at least \( n_0 m \) and less than \( (n_0 + 1)m \) zeros of \( f \). The same is true about the poles of \( f \).

(ii) Similarly as in (i) we can find unique \( n_1, n_2 \in \mathbb{Z} \) such that

\[
|q|^{n_1+1} < r_1 \leq |q|^{n_1} < |q|^{n_2} < r_2 \leq |q|^{n_2-1}.
\]

Hence

\[
(r_1, r_2) = (r_1, |q|^{n_1}) \cup \left( \bigcup_{k=n_1}^{n_2-1} (|q|^k, |q|^{k+1}) \right) \cup (|q|^{n_2}, r_2).
\]

Every annulus of the form \( \{z : |q|^{k+1} < |z| \leq |q|^k\} \), where \( k \in \mathbb{Z} \), contains equal amount of zeros and poles of \( f \) counted according to their multiplicities (we have denoted this number by \( m \)). Therefore the difference between the numbers of zeros and poles of \( f \) in the annulus \( \{z : r_1 < |z| < r_2\} \) is no greater than \( 2m \) because of the choice of \( n_1, n_2 \).

(iii) Let \( a_1, a_2, ..., a_m \) and \( b_1, b_2, ..., b_m \) be the zeros and the poles of \( f \) in \( \{z : |q| < |z| \leq 1\} \) respectively. Then all the zeros of \( f \) have the form \( a_{\mu, k} = a_kq^\mu \), where \( \mu \in \mathbb{Z}, k = 1, 2, ..., m \).
The same is true about the poles of $f$, namely $\beta_{\nu,k} = b_kq^n$, where $\nu \in \mathbb{Z}$, $k = 1, 2, ..., m$. So, $\frac{a_j}{b_kq^n} = \frac{a_j}{b_kq^n}$, where $l \in \mathbb{Z}$.

It is necessary to show that there exists $C > 0$ such that the inequality

$$\left| \frac{a_j}{b_kq^n} - 1 \right| > C$$

holds for all $j, k \in \{1, 2, ..., m\}$, and $l \in \mathbb{Z}$.

Suppose that for any $\varepsilon > 0$ there exist $j, k \in \{1, 2, ..., m\}$, and $l \in \mathbb{Z}$ such that

$$\left| \frac{a_j}{b_kq^n} - 1 \right| \leq \varepsilon.$$  \hspace{1cm} (10)

Without loss of generality we can assume that $|l| \leq 2$. Indeed, taking into account where $a_j, b_k$ belong to, we have

$$\left| \frac{a_j}{b_kq^n} \right| \leq \frac{1}{|q|} |q|^l \leq |q|, \quad l \geq 2.$$

Similarly,

$$\left| \frac{a_j}{b_kq^n} \right| \geq |q||q|^l \geq \frac{1}{|q|}, \quad l \leq -2.$$

So, for all $j, k \in \{1, 2, ..., m\}$, and $l \geq 2$

$$\left| \frac{a_j}{b_kq^n} - 1 \right| \geq 1 - |q|,$$

and for $l \leq -2$

$$\left| \frac{a_j}{b_kq^n} - 1 \right| \geq \frac{1}{|q|} - 1.$$

Let now $|l| < 2$. Choose

$$\varepsilon = \frac{1}{2} \min\{|a_jq^n - b_k| : j, k \in \{1, 2, ..., m\}, -1 \leq l \leq 1\}.$$

Then (10) implies

$$|a_jq^n - b_k| \leq \varepsilon|b_k| \leq \varepsilon.$$

That is

$$|a_jq^n - b_k| \leq \frac{1}{2} \min\{|a_jq^n - b_k| : j, k \in \{1, 2, ..., m\}, -1 \leq l \leq 1\}$$

which gives a contradiction.

(iv) We remind that $f$ has representation (7). It can be rewritten as follows

$$f(z) = cz^v \prod_{k=1}^{m} \left( \prod_{n=0}^{+\infty} \left( 1 - \frac{q^n z}{a_k} \right) \prod_{n=1}^{+\infty} \left( 1 - \frac{q^n a_k}{z} \right) \right), \quad z \in \mathbb{C}^*.$$  \hspace{1cm} (11)

Clearly, we can assume $c \neq 0$. Consider the integral means $I(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| d\theta$, $r > 0$. 
Let $z = re^{i\theta}$. We have for $r > 1$ [4, p. 8]
\[
\frac{1}{2\pi} \int_0^{2\pi} \log \left| 1 - \frac{z}{a_j} \right| \, d\theta = \log^+ \frac{r}{|a_j|},
\]
and, if $|a_j| \leq 1$
\[
\frac{1}{2\pi} \int_0^{2\pi} \log \left| 1 - \frac{a_j}{z} \right| \, d\theta = 0.
\]
The same is true for $b_j$.

Since for every $k \in \{1, 2, ..., m\}$ we have $|c_kq^{-n}| > 1$ for $n \in \mathbb{N}$, and $|c_kq^n| \leq 1$ for $n \in \mathbb{N} \cup \{0\}$, where $c_k$ is a zero or pole of $f$, then (11) implies
\[
I(r) = v \log r + \sum_{|a_j| > 1} \log^+ \frac{r}{|a_j|} - \sum_{|b_j| > 1} \log^+ \frac{r}{|b_j|} + \log |c|, \quad r > 1.
\]
Similarly, for $0 < r \leq 1$ we obtain
\[
I(r) = v \log r + \sum_{|a_j| \leq 1} \log^+ \frac{|a_j|}{r} - \sum_{|b_j| \leq 1} \log^+ \frac{|b_j|}{r} + \log |c|.
\]
Hence,
\[
\mathfrak{M}_v(r) = \frac{1}{|p|^{\mu(r)}} \frac{1}{|c|} \exp I(r) = \frac{1}{|c|} \exp \{ I(r) - \mu(r) \log |p| \}, \quad r > 0.
\]
Since $I(r)$ is convex with respect to $\log r$ and consequently continuous, $I(r)$ is bounded on $[|q|, 1]$. It follows from the definition of a $p$-loxodromic function of multiplicator $q$ that
\[
I(|q|^kr) = I(r) + k \log |p|
\]
for every $k \in \mathbb{Z}$. On the other hand
\[
\mu(|q|^kr) = \left[ \frac{k \log |q| + \log r}{\log |q|} \right] - 1 = k, \quad |q| \leq r < 1.
\]
That is
\[
\mathfrak{M}_v(|q|^kr) = \mathfrak{M}_v(r), \quad |q| \leq r < 1
\]
for all $k \in \mathbb{Z}$. Then we conclude that $\mathfrak{M}_v(r)$ remains bounded for all $r > 0$ which completes the proof.

\[\Box\]

4 Julia Exceptionality

**Definition 3.** Let $f_n, n \in \mathbb{N}$, be meromorphic functions in a domain $G$. A sequence $\{f_n(z)\}$ is said to be uniformly convergent to $f(z)$ on $G$ in the Carathéodory-Landau sense [1] if for any point $z_0 \in G$ there exists a disk $K(z_0)$ centered at this point such that $K(z_0) \subset G$ and
\[
(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall z \in K(z_0)) : |f_n(z) - f(z)| < \varepsilon,
\]
whenever $f(z_0) \neq \infty$, or
\[
\left| \frac{1}{f_n(z)} - \frac{1}{f(z)} \right| < \varepsilon,
\]
whenever $f(z_0) = \infty$. 
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Note that this convergence is equivalent to the convergence in the spherical metric.

**Definition 4.** A family $\mathcal{F}$ of meromorphic in $\mathbb{C}^*$ functions is said to be normal if every sequence $\{f_n\} \subseteq \mathcal{F}$ contains a subsequence which converges uniformly in the Carathéodory-Landau sense.

**Definition 5.** A meromorphic in $\mathbb{C}^*$ function $f$ is called Julia exceptional (see [7]) if for some $q$, $0 < |q| < 1$, the family $\{f_n(z)\}$, $n \in \mathbb{Z}$, where $f_n(z) = f(q^n z)$, is normal in $\mathbb{C}^*$.

In $\mathbb{C}$ there are few simple examples of Julia exceptional functions. But in $\mathbb{C}^*$ we have the following.

Let $f \in \mathcal{L}_{qp}$. We have

$$f_n(z) = f(q^n z) = p^n f(z)$$

for every $z \in \mathbb{C}^*$.

If $|p| > 1$, then a limiting function of the family $\{f_n(z)\}$, $n \in \mathbb{Z}$, is $\infty$. Otherwise, if $|p| < 1$, then a limiting function is 0. If $|p| = 1$, that is $p = e^{i\alpha}$, we have $f_n(z) = e^{i n\alpha} f(z)$. Hence, the set of limit functions depends on $\alpha$. If $\alpha = \frac{m\pi}{k}$, where $m \in \mathbb{Z}$, $k \in \mathbb{N}$, the number of limiting functions is less than or equals to $2k$. Otherwise, if $\alpha = \pi r$, where $r \in \mathbb{R} \setminus \mathbb{Q}$, the number of limiting functions is infinite.

**Example 2.** Let $f \in \mathcal{L}_{q^{\alpha}}$ with $\alpha = \frac{\pi}{4}$. Then

$$f_n(z) = f(q^n z) = p^n f(z) = e^{in\frac{\pi}{4}} f(z).$$

Thus, we obtain eight limiting functions

$$\pm f, \pm i f, \left(\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}\right) f, \left(-\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}\right) f.$$

Hence, $f$ is Julia exceptional in $\mathbb{C}^*$.

These results can be summarized as follows.

**Theorem 4.** Each function $f \in \mathcal{L}_{qp}$ is Julia exceptional in $\mathbb{C}^*$.

**References**


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Досліджується клас $p$-локсодромних функцій (мероморфних функцій, що задовольняють умову $f(qz) = pf(z)$ при деяких $q \in C \setminus \{0\}$ для всіх $z \in C \setminus \{0\}$). Доведено, що кожна $p$-локсодромна функція є Жюліа винятковою. Подано зображення таких функцій та описано розподіл їх нулів та полюсів.

Ключові слова і фрази: $p$-локсодромна функція, первинна функція Шотткі-Кляйна, Жюліа винятковість.